Business cycle analysis with short time series: a stochastic versus a non-stochastic approach

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Abstract

The idea of modelling of business cycle fluctuations using an autoregressive model with complex roots in its characteristic polynomial is well established in econometric analysis. It can be shown that such an approach induces a spectral density function with concentration of mass around a value (a frequency) related to a complex root. The resulting stochastic process is often referred to as a stochastic cycle model. However, there exists an alternative approach where business cycles are generated by time-varying mean which is driven by a deterministic function (a deterministic cycle approach). It is often claimed that the deterministic cycle is not sufficient for description of actual economic fluctuations due to its inflexibility. However, it is not clear that it is true for short series of data available for e.g. the emerging economies. In order to deal with the problem we consider a general model that encompasses both mechanisms (that of stochastic cycle and that of deterministic cycle). We illustrate applicability of the model analysing Polish data on GDP growth rates.

Keywords: *business cycles, time-series analysis, Flexible Fourier Form, empirical macroeconomics*

JEL Classification: C22, E32

1 Introduction

A concept of stochastic cycle, used as a tool for investigation of business cycle fluctuations, is considered in the literature (see for instance: Harvey and Trimbur, 2003; Trimbur, 2006; Koopman and Shephard, 2015; Pelagatti, 2016). For such a stationary univariate process its autocovariance function decays to zero with some "pseudo-period". It is often assumed that the cyclical component is a stationary ARMA process with complex conjugate roots in the characteristic polynomial. An extension to a multivariate model with cyclical fluctuations was considered in Azevedo et al. (2006), see also a trivariate example in Harvey et al. (2007). Koopman and Azevedo (2008) consider a multivariate model with stationary multiple cyclical process with common frequency at each coordinate.

On the other hand, one might consider its deterministic counterpart using a tool similar to the Flexible Fourier Form; see Gallant (1981), though the approach is less popular in the applied macro literature. For example Harvey (2004) considers y_t of the form:

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$$y_t = a\cos(\lambda_c t) + b\sin(\lambda_c t) + \varepsilon_t, \tag{1}$$

where λ_c is a frequency related to a cycle with period length of $2\pi/\lambda_c$; a and b are parameters and $\{\varepsilon_t\}$ represents a white noise process. The right-hand side of (1) can be interpreted as a restricted Flexible Fourier Form consisting of just one component. In general the deterministic model (1) is considered to be insufficient for description of actual business cycle fluctuations due to its inflexibility. However, it might be useful for analysis of short time-series that are available for emerging economies. To verify that we consider a model that encompasses the two concepts mentioned above. Note that a more general form of model (1) with multiple frequencies was considered in a nonparametric framework in Lenart and Pipień (2013a), Lenart and Pipień (2013b); Lenart and Pipień (2015) consider the idea of a deterministic cycle in examining the empirical properties of credit and equity cycle.

2 Stochastic cycles within an AR(p) model

Consider a general and zero-mean autoregressive process of order p, denoted by AR(p):

$$\Psi(L)y_t = \varepsilon_t, \tag{2}$$

with polynomial $\Psi(L) = 1 - \theta_1 L - \theta_2 L^2 - ... - \theta_P L^P$ and a white noise process $\{\varepsilon_t\}$ with variance σ_ε^2. Let $\alpha_1, \alpha_2, ..., \alpha_{p_1}$ be a set of distinct values taken by the real roots, with multiplicities denoted by $k_1, k_2, ..., k_{p_1}$. By $\beta_1, \beta_2, ..., \beta_{p_2}$ we denote the vector of all (different) values of complex roots with multiplicities denoted by $l_1, l_2, ..., l_{p_2}$. Hence,

$$\Psi(L) = \left(\prod_{i=1}^{p_1}(1 - L/\alpha_i)^{p_i}\right)\left(\prod_{i=1}^{p_2}[(1 - L/\beta_i)(1 - L/\overline{\beta}_i)]^{l_i}\right).$$

The spectral density function of the process (defined on the interval (0, π)) is of the form:

$$f(\lambda) = \sigma_\varepsilon^2(2\pi)^{-1}\left(\prod_{i=1}^{p_1}|1 - e^{-i\lambda}/\alpha_i|^{2p_i}\right)^{-1}\left(\prod_{i=1}^{p_2}[(1 - e^{-i\lambda}/\beta_i)(1 - e^{i\lambda}/\overline{\beta}_i)]^{l_i}\right)^{-1}. \tag{3}$$

Cyclical behaviour of the above processes is linked with a strong concentration of the spectral density in a neighbourhood of some frequency λ_0 (see for example Pelagatti, 2016). This frequency is one of the main quantities characterizing properties of cyclical fluctuations in such model.

Recall from the basic literature (see for example Brockwell and Davis, 2002) that for a simple AR(1) model with $\theta_1 < 0$, the spectral density function $f(\lambda)$ is an increasing function on (0, π), while for $\theta_1 > 0$, $f(\lambda)$ is decreasing function on (0, π). For an AR(2) model with two real roots, $f(\lambda)$ has local extremes at points: 0, π and might have one minimum in the interval (0,π), while in the case of two complex (conjugate) roots $f(\lambda)$ reaches one maximum in the interval (0,π) (see Brockwell and Davis, 2002). Therefore, in order to obtain a spectral density function with mass concentration in neighbourhood of some frequency one might consider autoregressive models with complex (conjugate) roots. This idea is well known in the literature, see Harvey and Trimbur (2003), Trimbur (2006), Koopman and Shephard (2015) and many others.

Let us consider the following AR(2) model with complex roots:

$$\Psi_z(L)y_t = \varepsilon_t \tag{4}$$

where $\Psi_z(L) = (1-zL)(1-\bar{z}L)$ and $z \in \mathrm{Z}$. Using a polar form $z = \rho e^{i\omega}$ we obtain:

$$\Psi_z(L) = \Psi_{(\rho,\omega)}(L) = (1-2Re[z]L + |z|^2 L^2) = (1-2\rho\cos(\omega)L + \rho^2 L^2). \tag{5}$$

with associated spectral density function (defined on the interval (0,π)) of the form:

$$f_y(\lambda) = \sigma_\varepsilon^2 (2\pi)^{-1}(1-2\rho\cos(\lambda+\omega)+\rho^2)^{-1}(1-2\rho\cos(\lambda-\omega)+\rho^2)^{-1}. \tag{6}$$

This is a basic model used in empirical modelling of cyclical fluctuations – it was examined in the first part of the last century by Samuelson (1939). Some extension to the AR(3) model with lag polynomial $(1-\alpha L)(1-zL)(1-\bar{z}L)$, where α is real number and z is a complex number was examined by Geweke (1988, 2016).

3 Cyclical properties of AR(*p*) model with multiple complex roots

The spectral density function (5) reaches a maximum at $\lambda_0 = \arccos\left((1+\rho^2)\cos(\omega)/(2\rho)\right)$, see Pelagatti (2016), p. 185 for similar computations. We propose the following reparametrization: $\omega = \arccos(2\rho\cos(\tilde{\omega})/(1+\rho^2))$. Consequently, the spectral density function has the form:

$$f_y(\lambda) = \sigma_\varepsilon^2 (2\pi)^{-1}\left(1-2\rho\cos\left(\lambda + g(\rho,\tilde{\omega})\right)+\rho^2\right)^{-1}\left(1-2\rho\cos\left(\lambda - g(\rho,\tilde{\omega})\right)+\rho^2\right)^{-1}, \tag{7}$$

with $g(\rho,\tilde{\omega}) = \arccos(2\rho\cos(\tilde{\omega})/(1+\rho^2))$. The spectral density function $f_y(\lambda)$ has a maximum at $\lambda_0 = \tilde{\omega}$. Note that another univariate cyclical process considered in the empirical economic literature, an ARMA(2,1) process with complex roots in the AR part, has spectral density function that differs from (6). The most important common properties of these two cyclical processes is that both spectral densities have maxima at known points. Note that for (6), $\lim_{|\rho|\to 1} f_y(\tilde{\omega}) = \infty$ which means that ρ determines the concentration of the spectral mass in the neighbourhood of frequency $\tilde{\omega}$.

In order to concentrate the spectral mass in the neighbourhood of the frequency of interest (with fixed ρ), Trimbur (2006) considers so-called *n*-th order cycle. In this paper we introduce an approach that is somewhat simpler. Our concept follows from purely mathematical properties of the spectral density function of an AR(*p*) process. Consider the general form:

$$\Psi^n_{(\rho,\tilde{\omega})}(L) y_{t,n} = \varepsilon_t. \tag{8}$$

Assuming $|\rho| < 1$, the above model is a stationary and inventible AR(2*n*) process. The spectral density function has the following form:

$$f_{y_n}(\lambda) = \sigma^2_\varepsilon (2\pi)^{-1} \left(1 - 2\rho\cos(\lambda + g(\rho,\tilde{\omega})) + \rho^2\right)^{-n} \left(1 - 2\rho\cos(\lambda - g(\rho,\tilde{\omega})) + \rho^2\right)^{-n}, \tag{9}$$

and it has a maximum at $\lambda_0 = \tilde{\omega}$, which means that the mass is concentrated in the neighbourhood of frequency $\tilde{\omega}$. Note that for $n\to\infty$ this maximum tends to infinity. Moreover, higher values of *n* imply stronger concentration around the maximum. The above concept is used here to introduce a stochastic cycle with known properties, controlled directly by model parameters.

4 A final model with stochastic and deterministic cycle

Finally we consider an AR(*p*) model with a Flexible Fourier-type time-varying mean μ_t:

$$(1 - \phi_1 L)(1 - \phi_2 L)\Psi^n_{(\rho,\tilde{\omega})}(L)(y_t - \mu_t) = \varepsilon_t, \tag{10}$$

where $\varepsilon_t \sim iiN(0, \sigma^2_\varepsilon)$ and

$$\mu_t = \delta_0 + a\sin(t\zeta) + b\cos(t\zeta), \tag{11}$$

and the latter corresponds to an almost periodic function with one frequency. We assume: $\delta_0, a, b \in R$, $\zeta, \tilde{\omega} \in (0,\pi)$, $-1 < \phi_i < 1$ for i = 1,2, $0 < \rho < 1$. The function μ_t reflects the deterministic cycle (1), while the polynomial $\Psi^n_{(\rho,\tilde{\omega})}(L)$ is related to the idea of stochastic cycle. The part $(1 - \phi_1 L)(1 - \phi_2 L)$ is related to the aspects of the dynamics of y_t that are not directly connected with its cyclical properties. Note that the spectral density function corresponding to the element $(1 - \phi_1 L)(1 - \phi_2 L)$ has a mass concentration around frequency 0 or π or around both. Parameters of the autoregressive part of the models are functions of $\tilde{\omega}$, ρ, ϕ_1 and ϕ_2.

The model is estimated using Bayesian techniques. As to the prior specification, we assume independent uniform priors for ζ, $\tilde{\omega}$, ρ, ϕ_1 and ϕ_2. As to δ_0, a, b, the priors are independent Student-*t*, and precision of the error process is a priori distributed as Gamma[3]. The estimation is undertaken using MCMC techniques (a random-walk Metropolis-Hastings algorithm). The inference is relatively easy as we consider only one frequency in the deterministic part. Bayesian estimation of deterministic cycle models with many frequencies is more complicated, as discussed by Lenart and Mazur (2016), see also Lenart et al. (2016).

5 Real data example

For the sake of illustration of the above concepts we analyse a series of quarterly growth rates of Polish GDP ([%], y-o-y, seasonally adjusted, 1999Q1-2015Q4, 68 observations)[4].

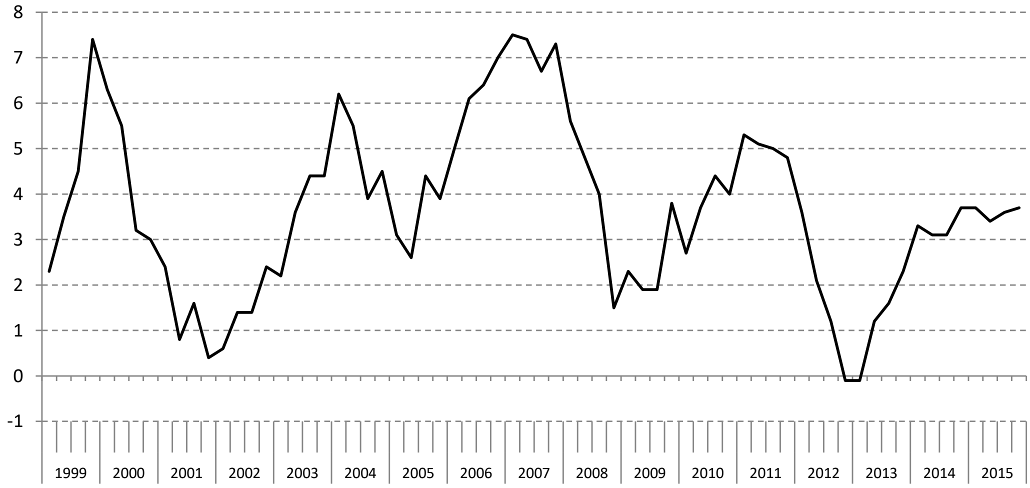

Fig. 1. GDP (y-o-y percentage change, quarterly data) from 1999Q1 to 2015Q4 (Poland).

In what follows we consider a sequence of models, beginning with a simple model with stochastic cycle only. It is then augmented as the deterministic cyclical component μ_t is introduced. Moreover, additional autoregressive parameters ϕ_1 and ϕ_2 (generating non-oscillatory behaviour) are also considered. For the additional autoregressive parameters we consider two cases with different inequality restrictions. The restrictions rule out negative values of parameters. In the general case there are two parameters taking values in the frequency domain that control the cyclical properties of the model: ζ for the deterministic

[3] Details of the prior specification and the estimation procedure are not elaborated here due to the space constraints. However, the details are available from the authors upon request.
[4] The data source is CEIC database accessed in March 2016.

cycle and $\tilde{\omega}$ for the stochastic cycle. Moreover, we assume that n = 2; A summary of specifications under consideration is given in Table 1.

Case no., *p*	**Stochastic cycle**	**Deterministic cycle**	**Real roots**	**Complex roots**
1, p = 6	Yes	No	$-1<\phi_i<1$	2
2, p = 6	Yes	No	$0<\phi_i<1$	2
3, p = 4	Yes	Yes	No	2
4, p = 6	Yes	Yes	$0<\phi_i<1$	2
5, p = 6	Yes	Yes	$-1<\phi_i<1$	2

Table 1. Models under consideration.

Posterior means for model components representing stochastic and deterministic cycles are reported in Fig.2. Shapes of marginal posterior distributions for selected model parameters are given in Fig. 3 (for all the cases in the figure priors are uniform). Moreover, Table 2 includes basic characteristics of posterior distributions for the standard deviation of the error term. Here we do not make an attempt to conduct a formal comparison of the nested cases. This is mostly because short macroeconomic series are not very informative so results of the formal comparison are seriously affected by prior assumptions.

Analysis of Fig. 2 suggests that the dynamic behavior of Polish GDP (in terms of the growth cycle) includes at least two different components. The first one represents regular fluctuations with period of approximately four years. It is captured by the stochastic part in the cases 1, 2 and 4 and by the deterministic part in cases 3 and 5. The other component is less regular and represents much longer cycles (lower frequencies). It is captured by the deterministic mechanism in case 4 and by the stochastic part in cases 3 and 5. The first component seems to be the dominant one.

It is interesting to ask which of the two components mentioned above is more likely to correspond to the stochastic cyclical mechanism. Interestingly, it seems to depend on the non-complex roots of the characteristic polynomial. If the roots are unrestricted and the deterministic component is present in the model, it takes over the fluctuations with period of approximately four years. However, if negative values of the parameters are ruled out (as in cases 2 and 4), the fluctuations are pinned down by the stochastic mechanism, while the deterministic part represents cycles characterized by longer period.

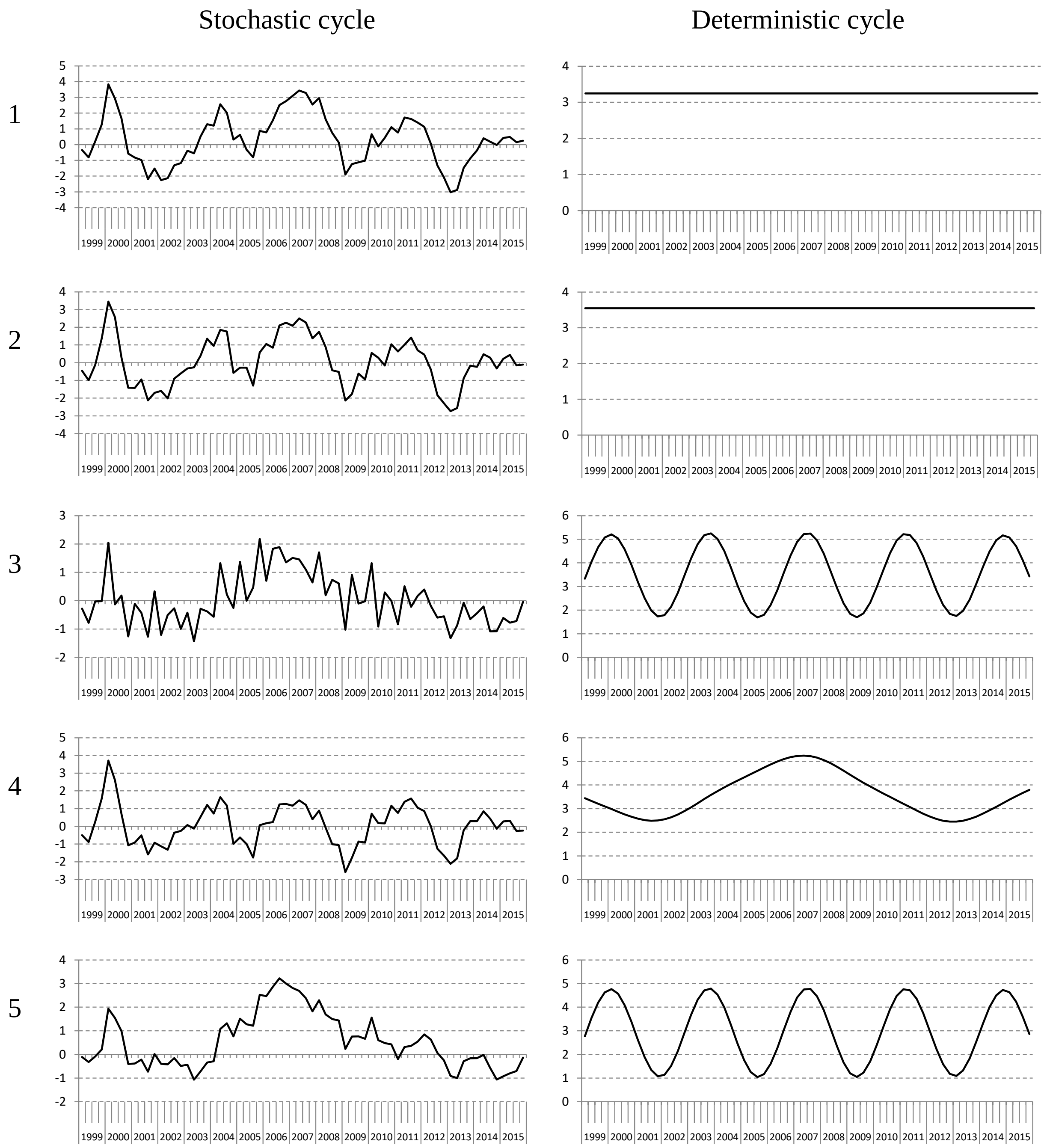

Fig. 2. Posterior means for (conditional) stochastic and deterministic components.

Similar effect is visible in Fig. 3, where higher values of ρ, corresponding to stochastic cycles with larger "amplitudes" can be seen in cases 2 and 4. The parameter $\tilde{\omega}$ is never separated from its lower bound. The posterior distribution of ζ in case 4 is bimodal, though the dominant mode represents fluctuations with longer period.

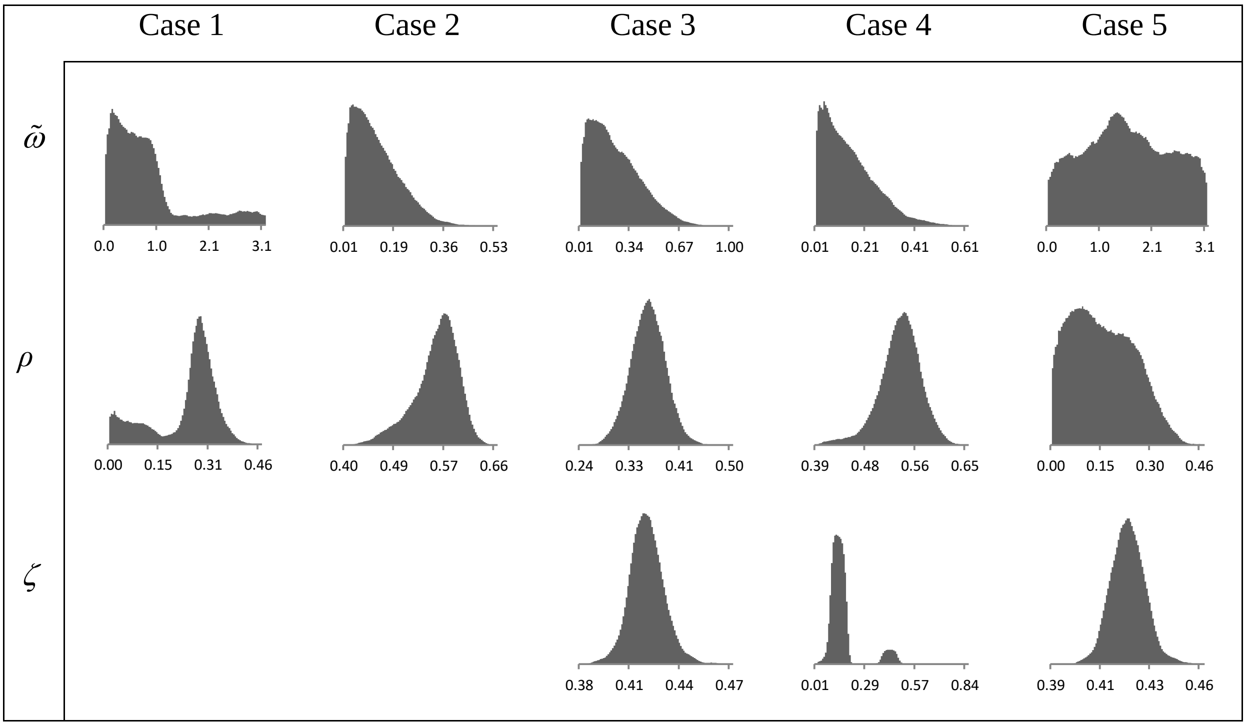

Fig. 3. Marginal posterior distributions for selected model parameters.

	Case 1	**Case 2**	**Case 3**	**Case 4**	**Case 5**
mean	0.983	1.234	1.131	1.007	0.762
standard deviation	0.184	0.235	0.219	0.185	0.146

Table 2. Characteristics of posterior distribution for square root of σ_{ε}^{2}.

Analysis of Table 2 suggests that the most general specification (case 5) is characterized by the lowest standard deviation of the error term. This informally confirms empirical relevance of the general model considered here.

In general the model proposed here leads to empirical conclusions that are in line with other research results as to Polish business cycle fluctuations. A summary of empirical results (based on the industrial production index) is given by Lenart et al. (2016). In particular it seems that for the series at hand both the stochastic and the deterministic mechanisms are empirically relevant. Moreover, the decomposition of the GDP dynamics seems to be sensitive to various aspects of model assumptions.

Conclusions

As to theoretical contribution of the paper, we develop a univariate time-series model that encompasses two approaches to modelling of macroeconomic cyclical fluctuations that are present in the applied macro literature. In one approach the cycle in mean is approximated using a deterministic concept that can be interpreted in terms of the Flexible Fourier Form. Within the other approach cycles are driven by complex roots of the characteristic polynomial of a stochastic AR(p) process. We impose restrictions on the AR part of the process in order to ensure that the spectral density function of the autoregressive part has maximum at a given point in the frequency domain. Additional equality restrictions are imposed in order to ensure concentration of the mass around that value. We also introduce parameters corresponding to real roots. The resulting model has two distinct sources of cyclical fluctuations with two (potentially different) frequencies, which provides an advantage compared to models that are most often used in the applied work (often allowing for just one frequency).

As to the empirical part, we illustrate properties of the model analysing Polish data on GDP dynamics. The purpose is to check whether the deterministic cycle is relevant for the countries for which only short series are available (like the emerging economies). The empirical results we obtain are in line with other analyses of Polish business cycle fluctuations. Moreover, both the stochastic and the deterministic model components seem to be empirically important. Verification of the out-of-sample performance (or predictive consequences) of various nested cases is left for further research.

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