Exponential smoothing models with time-varying periodic parameters

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Abstract

In the literature there is a growing number of models with time-varying in time parameters. The purpose of this article is to show the reduced form for the linear innovations model with time-varying periodic in time parameters. It will be show that when the state variables are eliminated from a linear innovations state space model with time-varying periodic in time parameters, an Periodic Autoregressive Integrated Moving Average model (PARIMA in short) with equality restrictions on parameters is obtained. This is the generalization for reduced form of the state space model with constant in time parameters. In particular, known models called are generalized to the periodic case. Finally, the real data example with macroeconomic data will be presented where the performance of competing models (based on Logarithmic Score) in pseudo-real forecasting exercise is used to assess the adequacy of a specific model.

Keywords: *state space model, exponential smoothing model, periodic ARIMA model*

JEL Classification: C22, E32

1 Introduction

Time-varying with the seasons sample autocovariance function are refereed to periodic time series. Such class of models called periodic models were introduced firstly by Hannan (1955), while in Gladyshev (1961) the Periodically Correlated time series with period T (PC(T) in short) ware defined and examined. For theory and applications for PC(T) time series see to Hurd and Miamee (2007). There are a few alternative ways to consider periodic (or seasonal) in time dynamic of parameters in time series models. The most popular is the usual ARMA model with periodic coefficients (PARMA in short). This generalization assumes periodic in time coefficients in AR and MA part, with the same period. Under appropriate regularity conditions the PARMA model is PC. The PARMA models are well examined in time and frequency domain (see for example Wyłomańska (2008)). In the same way the PARIMA and Seasonal PARIMA models can be defined. The alternative and more sophisticated models with time-varying periodic parameters are known. The applications and theoretical background concerning models with time-varying parameters can be found in Pagano (1978), Osborn (1991), Franses and Paap (2004), Burridge and Taylor (2001) and many others. The seasonal volatility (or seasonal heteroscedasticity) ware considered by Trimbur and Bell (2012), Berument and Sahin (2010), Doshi et al. (2011), Lenart (2017), and many other studies. In

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Novales and Fruto (1997) the authors consider performance of periodic models over non-periodic models in forecasting. Extension of Unobserved Component Models to periodic case ware considered in Koopman and Oomos (2008), Proietti (2004) and many others.

In this paper we consider the case of periodic coefficients in linear innovations state space model. The constant parameters were examined in details in Hyndman et al. (2008). Note that modeling with periodic coefficients is effective if appropriate models and estimations procedures are considered. Additionally, good in-sample fit is not equivalent with good forecasting power. We decide to extend this class of time series to periodic case since the components of exponential smoothing models (e.g. level component, trend component, seasonal component) has an natural interpretation. By assuming periodicity for parameters the extended natural interpretation is possible (for example seasonal conditional volatility). In the first part we examine the exponential smoothing models with periodic coefficients from theoretical point of view. We show the exact reduced PARIMA forms for chosen models. In empirical part, we would like to shed light on the forecasting problem with such class of time series. We use data concerning monthly price in education COICOP in Poland (Jan. 1997 to Dec. 2015).

2 Reduced Form for the General Linear Innovations Model with periodic coefficient

We consider the following general linear innovation model of the form:

$$\begin{cases} y_t = \boldsymbol{\omega}'\mathbf{x}_{t-1} + \varepsilon_t & \text{observation equation} \\ \mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{g}_t\varepsilon_t & \text{state equation,} \end{cases} \tag{1}$$

where y_t is real-valued observed time series and $\mathbf{x_t}$ is the state vector, $\varepsilon_t \sim IID(0,\sigma_t^2)$. For the matrixes $\boldsymbol{\omega}$ and $\mathbf{F}$ we assume that are constant. For matrix $\mathbf{g}_t$ we assume that contains element being an periodic functions with the same period $T > 1$. The periodicity of variance σ_t^2 is also assumed. Hyndman et al. (2008) consider case with constant $\mathbf{g}_t$, σ_t^2 and proved that such model has reduced ARIMA form. In the next part we show that generalization to periodic case produce reduced periodic ARIMA model with period T, i.e. ARIMA model with time-varying periodic coefficients with period T.

Based on the same steps as in Hyndman et al. (2008) (page 170, we drop the details) the general linear innovation model can be reduced to Periodic ARIMA model of the general form:

$$\eta(L)y_t = \theta_t(L)[\varepsilon_t], \tag{2}$$

where the autoregressive polynomial equals

$$\eta(L) = \det(I - \mathbf{F}L) \tag{3}$$

and time-varying moving average polynomial equals

$$\theta_t(L) = \boldsymbol{\omega}' \operatorname{adj}(I - \mathbf{F}L)L[\mathbf{g}_t] + \det(I - \mathbf{F}L), \tag{4}$$

and $L^k[X_t] = X_{t-k}$ for any $k, t \in \mathbb{Z}$. Note that the polynomial θ depends on t. Hence, after this polynomial we put the argument in brackets $[\cdot]$. To clarify, we consider the following local level model of the form

$$\begin{cases} y_t = l_{t-1} + \varepsilon_t \\ l_t = l_{t-1} + \alpha_t \varepsilon_t, \end{cases} \tag{5}$$

where $\varepsilon_t \sim \mathrm{NID}(0, \sigma_t^2)$ and α_t, σ_t are periodic with period T (see Hyndman et al. 2008, page 40 in case of constant parameters). In such case, the elementary calculations give that $(1-L)y_t = (1 + (\alpha_{t-1} - 1)L)[\varepsilon_t]$. It means that this model has reduced PARIMA(0,1,1) model form with periodic variance of the white noise and equality restrictions on parameters.

In the next section we consider more advanced models with periodic parameters. We consider damped level model with seasonal pattern; local additive seasonal model with damped level and trend pattern and finally double damped local trend model. The extension of known seasonal $ARIMA(p,d,q)(P,D,Q)_m$ model with season m to periodic case (with period T) we denote by seasonal $PARIMA(p,d,q)(P,D,Q)_m$ with season m and period T. Note that it is natural to consider $T = m$.

2.1 Damped level model with seasonal pattern

In this section we consider model with damped level pattern and seasonal pattern with periodic coefficients of the form

$$\begin{cases} y_t = \phi l_{t-1} + s_{t-m} + \varepsilon_t \\ l_t = \phi l_{t-1} + \alpha_t \varepsilon_t \\ s_t = s_{t-m} + \gamma_t \varepsilon_t, \end{cases} \tag{6}$$

where m is the length of the season for seasonal pattern and α_t, γ_t, σ_t^2 are periodic functions at t with the same period $T > 1$. Equivalently, above model can be written in form

$$\begin{cases} y_t = \phi l_{t-1} + s_{t-m} + \varepsilon_t \\ l_t = \alpha_t (y_t - s_{t-m}) + (1 - \alpha_t)\phi l_{t-1} \\ s_t = \gamma_t (y_t - \phi l_{t-1}) + (1 - \gamma_t) s_{t-m}. \end{cases} \tag{7}$$

To simplify this section we assume that $T = m = 4$. But the general case is passible. In such specific case we have $\mathbf{x}_{t'} = (l_t \quad s_t \quad s_{t-1} \quad s_{t-2} \quad s_{t-3})$, $\boldsymbol{\omega}' = (\phi \quad 0 \quad 0 \quad 0 \quad 1)$, $\mathbf{g}_{t'} = (\alpha_t \quad \gamma_t \quad 0 \quad 0 \quad 0)$,

$$\mathbf{F} = \begin{pmatrix} \phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad I - \mathbf{F}L = \begin{pmatrix} 1-L\phi & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -L \\ 0 & -L & 1 & 0 & 0 \\ 0 & 0 & -L & 1 & 0 \\ 0 & 0 & 0 & -L & 1 \end{pmatrix}. \tag{8}$$

Elementary calculations give that $\eta(L) = \det(I - \mathbf{F}L) = (1 - L^4)(1 - L\phi)$ and

$$\mathrm{adj}(I - \mathbf{F}L) = \begin{pmatrix} 1-L^4 & 0 & 0 & 0 & 0 \\ 0 & 1-L\phi & L^3 - L^4\phi & L^2 - L^3\phi & L - L^2\phi \\ 0 & L - L^2\phi & 1-L\phi & L^3 - L^4\phi & L^2 - L^3\phi \\ 0 & L^2 - L^3\phi & L - L^2\phi & 1-L\phi & L^3 - L^4\phi \\ 0 & L^3 - L^4\phi & L^2 - L^3\phi & L - L^2\phi & 1-L\phi \end{pmatrix}. \tag{9}$$

Hence (after calculations) $\boldsymbol{\omega}'\mathrm{adj}(I - \mathbf{F}L)L[\mathbf{g}_t] = (-\phi\alpha_{t-5} - \phi\gamma_{t-5})L^5 + \gamma_{t-4}L^4 + \phi\alpha_{t-1}L.$ Finally

$$\theta_t(L) = \phi(1 - \alpha_{t-5} - \gamma_{t-5})L^5 + (\gamma_{t-4} - 1)L^4 + \phi(\alpha_{t-1} - 1)L + 1. \tag{10}$$

Hence, the considered model is an seasonal PARIMA $(1,0,5)(0,1,0)_4$ model with period $T = 4$, periodic variance of the white noise and equality restrictions on parameters.

2.2 Local additive seasonal model with damped level

In this section we consider additive model with damped level, seasonal pattern, and trend pattern of the form

$$\begin{cases} y_t = \phi_t l_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t \\ l_t = \phi l_{t-1} + b_{t-1} + \alpha_t \varepsilon_t \\ b_t = b_{t-1} + \beta_t \varepsilon_t \\ s_t = s_{t-m} + \gamma_t \varepsilon_t, \end{cases} \tag{11}$$

where we assume that α_t, β_t, γ_t and σ_t^2 are periodic functions at t with period $T = m$. For $T = m = 4$ we have $\mathbf{x}_t' = (l_t \quad b_t \quad s_t \quad s_{t-1} \quad s_{t-2} \quad s_{t-3})$, $\boldsymbol{\omega}' = (\phi \quad 1 \quad 0 \quad 0 \quad 0 \quad 1)$, $\mathbf{g}_t' = (\alpha_t \quad \beta_t \quad \gamma_t \quad 0 \quad 0 \quad 0)$,

$$\mathbf{F}=\begin{pmatrix}\phi & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix},\qquad I-\mathbf{F}L=\begin{pmatrix}1-L\phi & -L & 0 & 0 & 0 & 0\\ 0 & 1-L & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & -L\\ 0 & 0 & -L & 1 & 0 & 0\\ 0 & 0 & 0 & -L & 1 & 0\\ 0 & 0 & 0 & 0 & -L & 1\end{pmatrix}. \quad (12)$$

Note that $\eta(L)=\det(I-\mathbf{F}L)=(1-L)\left(L^5\phi-L^4-L\phi+1\right)=(1-L)(1-L^4)(1-\phi L)$. In addition,

$$\begin{aligned}\boldsymbol{\omega}'\operatorname{adj}(I-\mathbf{F}L)L[\mathbf{g}_t] &=\phi\left(\alpha_{t-6}+\gamma_{t-6}\right)L^6+\left(-\phi\alpha_{t-5}-\beta_{t-5}-\phi\gamma_{t-5}-\gamma_{t-5}\right)L^5\\ &+\gamma_{t-4}L^4-\phi\alpha_{t-2}L^2+\left(\phi\alpha_{t-1}+\beta_{t-1}\right)L,\end{aligned} \quad (13)$$

which means that

$$\begin{aligned}\theta_t(L) &=\phi\left(\alpha_{t-6}+\gamma_{t-6}-1\right)L^6+\left(\phi-\phi\alpha_{t-5}-\beta_{t-5}-\phi\gamma_{t-5}-\gamma_{t-5}+1\right)L^5\\ &+\left(\gamma_{t-4}-1\right)L^4+\phi\left(1-\alpha_{t-2}\right)L^2+\left(\phi\alpha_{t-1}+\beta_{t-1}-\phi-1\right)L+1,\end{aligned} \quad (14)$$

or equivalently

$$\begin{aligned}\theta_t(L) &=(1-L)\left[\phi\left(1-\alpha_{t-5}-\gamma_{t-5}\right)L^5+\left(\beta_{t-4}+\gamma_{t-4}-1\right)L^4+\beta_{t-3}L^3\right.\\ &\left.+\beta_{t-2}L^2+\left(\phi\alpha_{t-1}+\beta_{t-1}-\phi\right)L+1\right],\end{aligned} \quad (15)$$

which means that the factor $1-L$ is common in polynomials $\eta(L)$ and $\theta_t(L)$. Hence,

$$\eta(L)=L^5\phi_t-L^4-L\phi_t+1=(1-L^4)(1-\phi L), \quad (16)$$

$$\begin{aligned}\theta_t(L) &=\phi\left(1-\alpha_{t-5}-\gamma_{t-5}\right)L^5+\left(\beta_{t-4}+\gamma_{t-4}-1\right)L^4+\beta_{t-3}L^3\\ &+\beta_{t-2}L^2+\left(\phi\alpha_{t-1}+\beta_{t-1}-\phi\right)L+1.\end{aligned} \quad (17)$$

The considered model is an seasonal $\text{PARIMA}(1,0,5)(0,1,0)_4$ model with period $T=4$, periodic variance of the white noise and equality restrictions on parameters.

2.3 Double damped local trend model

Following by Hyndman et al. (2008), page 181 we consider

$$\begin{cases} y_t=\phi_1 l_{t-1}+\phi_2 b_{t-1}+\varepsilon_t\\ l_t=\phi_1 l_{t-1}+\phi_2 b_{t-1}+\alpha_t\varepsilon_t\\ b_t=\phi_2 b_{t-1}+\beta_t\varepsilon_t,\end{cases} \quad (18)$$

where α_t, β_t and σ_t^2 are periodic functions at t with period $T>0$. We have $\mathbf{x}_t'=\begin{pmatrix}l_t & b_t\end{pmatrix}$, $\boldsymbol{\omega}'=\begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}$, $\mathbf{g}_t'=\begin{pmatrix}\alpha_t & \beta_t\end{pmatrix}$,

$$\mathbf{F}=\begin{pmatrix}\phi_1 & \phi_2\\ 0 & \phi_2\end{pmatrix},\qquad I-\mathbf{F}L=\begin{pmatrix}1-L\phi_1 & -L\phi_2\\ 0 & 1-L\phi_2\end{pmatrix}. \quad (19)$$

Elementary calculations give that $\eta(L) = \det(I - \mathbf{F}L) = (1-\phi_1 L)(1-\phi_2 L)$. In addition,

$$\theta_t(L) \quad = \phi_1\phi_2(1-\alpha_{t-2})L^2 + (\phi_1\alpha_{t-1} + \phi_2\beta_{t-1} - \phi_1 - \phi_2)L + 1. \tag{20}$$

It means that the considered model is $\mathrm{PARMA}(2,2)$ model with period T, periodic variance of the white noise and equality restrictions on parameters.

3 Forecasting experiment

We consider monthly price in education in Poland (monthly rate of change, m-o-m, HICP (2015 = 100), source: Eurostat) form Jan. 1997 to Dec. 2015 (see Fig. 1). This price process is an important driver of inflation at September and October, where the peaks are observed (due to some administrative regulations). The seasonal pattern in obvious is such data (with period 12), while the trend is clearly not observed. Therefore, we propose to apply damped level model with seasonal pattern (see Section 2.1). We consider 9 different specifications for this model. For time-varying parameters: σ_t, γ_t, α_t we consider sequence of labels during one year (see details in Table 1). The same number at different months (for example 1 and 1 or 2 and 2) means the same value of parameter. Different numbers (for example 1 and 2 or 1 and 2 and 3) at months means that the values are different. For example, model M1 assumes constant parameters, while M3 assumes that σ_t and α_t are constant and γ_t has three values during year (labeled by: 1,2,3).

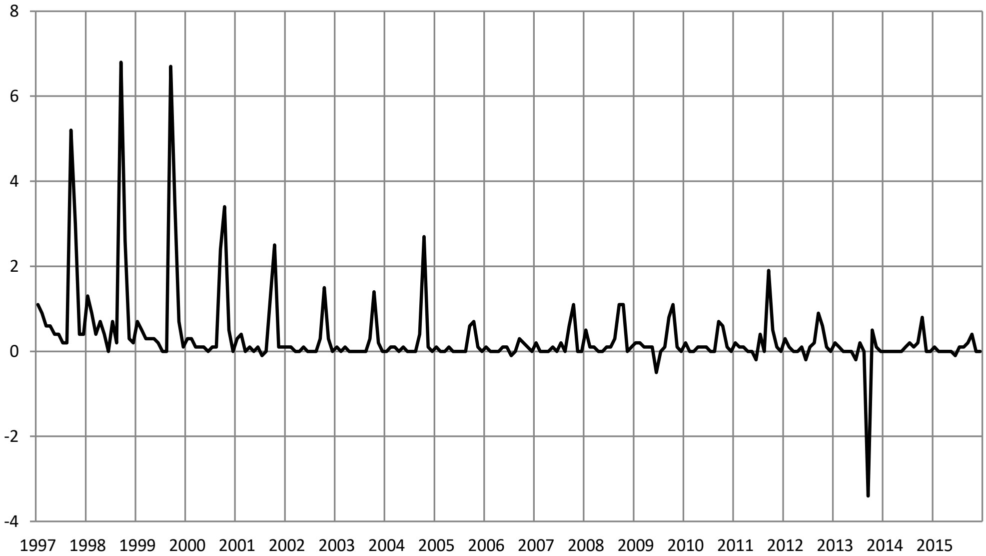

Fig. 1. Monthly rate of change for education COICOP (m-o-m percentage change, monthly data) from Jan. 1997 to Dec. 2015 (Poland).

		M1	M2	M3	M4	M5	M6	M7	M8	M9
	January	1	1	1	1	1	1	1	1	1
	February	1	1	1	1	1	1	1	1	1
	⋮									
	July	1	1	1	1	1	1	1	1	1
σ_t	August	1	1	1	1	1	1	1	1	1
	September	1	1	1	1	1	2	2	2	2
	October	1	1	1	1	1	2	3	2	3
	November	1	1	1	1	1	1	1	1	1
	December	1	1	1	1	1	1	1	1	1
	January	1	1	1	1	1	1	1	1	1
	February	1	1	1	1	1	1	1	1	1
	⋮									
	July	1	1	1	1	1	1	1	1	1
γ_t	August	1	1	1	1	1	1	1	1	1
	September	1	2	2	1	1	1	1	2	2
	October	1	2	3	1	1	1	1	2	3
	November	1	1	1	1	1	1	1	1	1
	December	1	1	1	1	1	1	1	1	1
	January	1	1	1	1	1	1	1	1	1
	February	1	1	1	1	1	1	1	1	1
	⋮									
	July	1	1	1	1	1	1	1	1	1
α_t	August	1	1	1	1	1	1	1	1	1
	September	1	1	1	2	2	1	1	2	2
	October	1	1	1	2	3	1	1	2	3
	November	1	1	1	1	1	1	1	1	1
	December	1	1	1	1	1	1	1	1	1

Table 1. Models parameters characteristics (under consideration).

We divide our sample into two parts: a training and a forecasting period. For the estimation we use 13-year rolling window. We start the estimation using data set up to Dec. 2009. After each new predictive distribution evaluation we add next observation and delete last one. Therefore, the length of the sample is constant over time (n=168). Finally, we collect sixty predictive distributions for 12 month ahead. For nowcasting the average logarithmic score was calculated (see Fig. 2). To compare logarithmic score for different models we use the test proposed by Amisano and Giacomini (2007). At significance level 5% in group of first five models we cannot reject null hypothesis that the logarithmic scores are different (comparison by pairs). The same conclusion concerns group of models M6-M9. Finally, if we compare any model from group M1-M5 with any model from group M6-M9 then we reject null hypothesis assuming equal logarithmic score, against alternative hypothesis that chosen model from group M6-M9 has higher logarithmic score. Summing up, the models with time-varying periodic in time variance of white noise improve forecasting performance over alternative models with constant variance of error term.

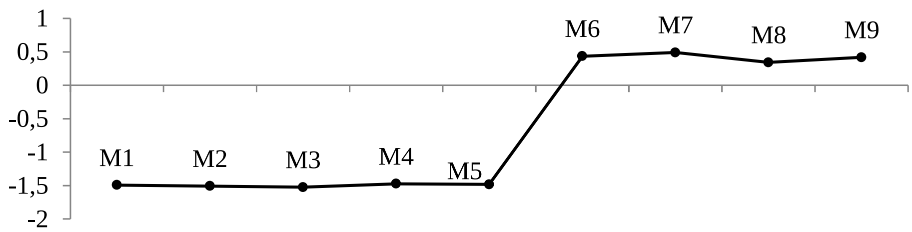

Fig. 2. Average logarithmic score for different considered models based on sixty predictive distributions (nowcasting).

Conclusions

In this paper we show the exact form for the linear innovations model with time-varying periodic in time parameters. It is Periodic Autoregressive Integrated Moving Average form with equality restrictions on parameters. Specific models ware considered with level, trend and seasonal pattern. The real data example was considered with monthly time series concerning price in education in Poland. We consider problem of forecasting performance based on logarithmic score rule. Our main findings from real data example is that in such case models with periodic in time variance of error term outperforms the considered models with constant variance of error terms. In addition, from statistical point of view (at significant level 5%) allowing seasonality only in smoothing parameters in level (α_t) and seasonal pattern (γ_t) doesn't produce better forecasting preference. Note that alternative model specifications should be considered and compared. Also the test proposed by Amisano and Giacomini (2007) should be adjust to the periodic case.

Acknowledgements

Publication was financed from the funds granted to the Faculty of Finance and Law at Cracow University of Economics, within the framework of the subsidy for the maintenance of research potential.

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