Generalization of the geo-logarithmic price index family

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Abstract

The paper presents a proposition of the geo-logarithmic index family generalization. It is shown that the Fisher and Walsh price indices are particular cases of this family and, under some assumptions, that the Törnqvist index can be its approximation. Due to the fact that the above-mentioned indices are superlative, they are best proxies for the Cost of Living Index (COLI). In practice, the Consumer Price Index (CPI), which is calculated by using the Laspeyres formula, is a measure of inflation but according to the economic approach in price index theory, a well-defined index should satisfy the Laspeyres-Paasche bounding test. In the simulation study we verify this test in the case of the generalized geo-logarithmic index family. The general conclusion is that values of almost all indices from the mentioned family are in the interval with bounds determined by the Laspeyres and Paasche price indices.

Keywords: *CPI, COLI, the Laspeyres index, geo-logarithmic price index family*

JEL Classification: E1, E2, E3

1 Introduction

The literature on axiomatic index theory is very wide (Krstcha, 1988; Balk, 1995). From a theoretical point of view, a well-constructed index should satisfy a group of postulates (tests) coming from the axiomatic index theory. A system of minimum requirements of an index comes from Marco Martini (1992b). According to the above-mentioned system a price index should satisfy at least three conditions: *identity*, *commensurability* and *linear homogeneity* (von der Lippe, 2007).

As it is known, all geo-logarithmic indices are proportional, commensurable and homogeneous, together with their cofactors (Martini, 1992a). Geo-logarithmic price indices satisfying the axioms of monotonicity, basis reversibility and factor reversibility have been investigated in the paper of Marco Fattore (2010). In the mention paper it is shown that the superlative Fisher price index does not belong to this family of indices. This paper presents a proposition of the geo-logarithmic index family generalization. It is shown that the Fisher and Walsh price indices are particular cases of this family and, under some assumptions, that the Törnqvist index can be its approximation. In the simulation study according to the economic approach in the price index theory we verify the Laspeyres-Paasche bounding test (von der Lippe (2007)) in the case of the generalized geo-logarithmic index family.

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2 Geo-logarithmic price index family

Let us consider a group of N commodities observed at times s, t and let us denote[2]:

$p_s = [p_{s1}, p_{s2}, ..., p_{sN}]'$ - a vector of prices at time s;

$p_t = [p_{t1}, p_{t2}, ..., p_{tN}]'$ - a vector of prices at time t;

$q_s = [q_{s1}, q_{s2}, ..., q_{sN}]'$ - a vector of quantities at time s;

$q_t = [q_{t1}, q_{t2}, ..., q_{tN}]'$ - a vector of quantities at time t.

Let us denote by $\tau(x, y)$ the logarithmic mean of two positive real numbers x and y, *i.e.*

$$\tau(x, y) = \frac{x - y}{\ln(x) - \ln(y)}, \tag{1}$$

if $x \neq y$ and $\tau(x, y) = x$ otherwise (Carlson, 1972).

For $x, y \in [0,1]$, let q^x and q^y be two vectors, whose components are defined as follows

$$q_i^x = q_{ti}^x q_{si}^{1-x}, \; q_i^y = q_{ti}^y q_{si}^{1-y}, \text{ for } i = 1,2,..., N \tag{2}$$

and let

$$w_{si}^x = \frac{p_{si} q_i^x}{\sum_{i=1}^{N} p_{si} q_i^x}, \tag{3}$$

$$w_{ti}^y = \frac{p_{ti} q_i^y}{\sum_{i=1}^{N} p_{ti} q_i^y}. \tag{4}$$

The geo-logarithmic, or the P_{xy}, family is the class of price indices defined by Fattore (2006)

$$P_{xy}(q_s, q_t, p_s, p_t) = \prod_{i=1}^{N} \left(\frac{p_{ti}}{p_{si}}\right)^{v_i^{xy}} \tag{5}$$

where weights v_i^{xy} are as follows

$$v_i^{xy} = \frac{\tau(w_{si}^x, w_{ti}^y)}{\sum_{j=1}^{N} \tau(w_{sj}^x, w_{tj}^y)}. \tag{6}$$

It is easy to verify that (Fattore, 2010)

$$P_{La} = \frac{\sum_{i=1}^{N} p_{ti} q_{si}}{\sum_{i=1}^{N} p_{si} q_{si}} = P_{00}, \tag{7}$$

[2] The time moment s is considered as the *basis*, i.e. the reference situation, for the comparison.

$$P_{Pa} = \frac{\sum_{i=1}^{N} p_{ti} q_{ti}}{\sum_{i=1}^{N} p_{si} q_{ti}} = P_{11}, \tag{8}$$

$$P_{W} = \frac{\sum_{i=1}^{N} p_{ti} \sqrt{q_{si} q_{ti}}}{\sum_{i=1}^{N} p_{si} \sqrt{q_{si} q_{ti}}} = P_{0.5\ 0.5} \tag{9}$$

where P_{La}, P_{Pa} and P_{W} denote the Laspeyres, Paasche and Walsh price indexes respectively. It is a very desirable observation that not only Laspeyres and Paasche indices (used in practice) are particular cases of the geo-logarithmic price index family but also the superlative Walsh price index. Nevertheless, as it was mentioned before, the most important, superlative Fisher price index does not belong to this family.

3 Generalization of the geo-logarithmic price index family

Similarly to (2), (3), (4) and (6) let us denote by

$$q_i^{Ax} = q_{ti}^{x} q_{si}^{1-x},\ q_i^{Ay} = q_{ti}^{y} q_{si}^{1-y}, \tag{10}$$

$$q_i^{Bx} = q_{ti}^{1-x} q_{si}^{x},\ q_i^{By} = q_{ti}^{1-y} q_{si}^{y}, \tag{11}$$

$$w_{si}^{Ax} = \frac{p_{si} q_i^{Ax}}{\sum_{i=1}^{N} p_{si} q_i^{Ax}},\ w_{ti}^{Ay} = \frac{p_{ti} q_i^{Ay}}{\sum_{i=1}^{N} p_{ti} q_i^{Ay}}, \tag{12}$$

$$w_{si}^{Bx} = \frac{p_{si} q_i^{Bx}}{\sum_{i=1}^{N} p_{si} q_i^{Bx}},\ w_{ti}^{By} = \frac{p_{ti} q_i^{By}}{\sum_{i=1}^{N} p_{ti} q_i^{By}}, \tag{13}$$

$$v_{Ai}^{xy} = \frac{\tau(w_{si}^{Ax}, w_{ti}^{Ay})}{\sum_{j=1}^{N} \tau(w_{sj}^{Ax}, w_{tj}^{Ay})},\ v_{Bi}^{xy} = \frac{\tau(w_{si}^{Bx}, w_{ti}^{By})}{\sum_{j=1}^{N} \tau(w_{sj}^{Bx}, w_{tj}^{By})}, \tag{14}$$

for $i = 1,2,..., N$, $x, y \in [0,1]$.

Under significations (10)-(14) we define the new class of price indices ($\widetilde{P}_{xyz}$) as follows

$$\widetilde{P}_{xyz} = \{\prod_{i=1}^{N} (\frac{p_{ti}}{p_{si}})^{v_{Ai}^{xy}}\}^{z} \{\prod_{i=1}^{N} (\frac{p_{ti}}{p_{si}})^{v_{Bi}^{xy}}\}^{1-z},\ z \in [0,1]. \tag{15}$$

Obviously, these two price indices (defined inside curly brackets in (15)) satisfy the Martini's minimal requirements. The first one is identical with P_{xy} index (for fixed values of x and y) and his axiomatic properties were proved by Fattore (2010). The proof of the same group of axioms in the case of the second price index (being inside curly bracket on the right side of (15)) would be analogous. Thus, since a formula (15) is a geometric mean of two general indices satisfying Martini's requirements, we can conclude that each index from $\widetilde{P}_{xyz}$ also satisfies *identity, commensurability* and *linear homogeneity* (Białek, 2012). Moreover, it holds that

$$\widetilde{P}_{xy1} = \widetilde{P}_{1-x\ 1-y\ 0} = P_{xy}\,, \tag{16}$$

$$\widetilde{P}_{001} = \widetilde{P}_{110} = P_{La}\,, \tag{17}$$

$$\widetilde{P}_{111} = \widetilde{P}_{000} = P_{Pa}\,, \tag{18}$$

$$\widetilde{P}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}} = P_W\,, \tag{19}$$

and, what is more interesting, the Fisher price index belongs to the considered class since

$$\widetilde{P}_{00\frac{1}{2}} = \widetilde{P}_{11\frac{1}{2}} = P_F\,. \tag{20}$$

The relation (16) means that the P_{xy} family is a special case of the $\widetilde{P}_{xyz}$ family. It can be also proved[3] that the following approximation holds

$$\forall i \in \{1,2,...,N\}\ \ q_{si} \approx q_{ti}\ \wedge\ w_{si}^{x} \approx w_{ti}^{y}\ \ \Rightarrow\ \ \widetilde{P}_{xyz} \approx P_T \tag{21}$$

where P_T denotes the Törnqvist price index defined as

$$P_T = \prod_{i=1}^{N} \left(\frac{p_{ti}}{p_{si}}\right)^{\frac{w_{si}^{0}+w_{ti}^{1}}{2}}\,. \tag{22}$$

4 Geo-logarithmic price index family and its generalization vs the Laspeyres-Paasche bounding test

The *Consumer Price Index* (CPI) is commonly used as a basic measure of inflation. The index approximates changes in the costs of household consumption assuming the constant utility (COLI, *Cost of Living Index*). In the so called economic approach, upper and lower bounds for the COLI are provided by the Laspeyres and Paasche price index formulas. These bounds are obtained under the assumption about cost minimizing behavior. If the price index value is within these bounds then we say that this price index satisfies the Laspeyres-Paasche bounding test belonging to the group of mean value tests (von der Lippe, 2007).

[3] The proof of this approximated will be presented in the extended version of the paper.

4.1 Simulation study

Let us take into consideration a group of $N = 12$ components, where prices and quantities are normally distributed as follows: $p_i^\tau \sim N(p_{i0}^\tau, v_i^\tau p_{i0}^\tau)$, $q_i^\tau \sim N(q_{i0}^\tau, u_i^\tau q_{i0}^\tau)$, where $\tau = s,t$, $N(\mu, \sigma)$ - denotes the normal distribution with the mean μ and the standard deviation σ, v_i^τ - denotes the volatility coefficient of the $i-$th price at time τ, i.e. $v_i^\tau = D(p_i^\tau)/p_{i0}^\tau$, u_i^τ - denotes the volatility coefficient of the $i-$th quantity at time τ, i.e. $u_i^\tau = D(q_i^\tau)/q_{i0}^\tau$. Before generating prices and quantities we generated values of volatility coefficients using uniform distributions, i.e. $v_i^\tau \sim U(0, v^\tau)$ and $u_i^\tau \sim U(0, u^\tau)$. Expected values of prices and quantities are described by following vectors

$$P_0^t = [1000,\ 1700,\ 500,\ 3.2,\ 105,\ 1150,\ 1000,\ 1600,\ 500,\ 4.2,\ 110,\ 1100]';$$

$$P_0^s = [900,\ 1600,\ 460,\ 3,\ 100,\ 1000,\ 900,\ 1530,\ 480,\ 4,\ 100,\ 1000]';$$

$$Q_0^t = [200,\ 200,\ 3000,\ 500,\ 340,\ 700,\ 800,\ 500,\ 3000,\ 500,\ 340,\ 700]';$$

$$Q_0^s = [350,\ 550,\ 5000,\ 710,\ 350,\ 890,\ 850,\ 600,\ 5000,\ 700,\ 550,\ 800]'.$$

In our experiment we are going to control values of volatility coefficients of prices and quantities by setting values of v^s, v^t, u^s, u^t. We consider here only two cases[4]: Case 1 (volatilities of price and quantity processes are small and the quantity response on price changes is quite normal); Case 2 (volatilities of prices and quantities are large, the quantity response on price changes is strongly fluctuated). For each case we generate values of vectors of prices and quantities in $n = 1000$ repetitions. Let us denote for fixed values of x and y and for each of $k-th$ repetition: $\Delta 1_k = (P_{xy} - \min(P_{La}, P_{Pa}))_k$, $\Delta 2_k = (\max(P_{La}, P_{Pa}) - P_{xy})_k$, $\Delta 3_k = (\widetilde{P}_{xy\frac{1}{2}} - \min(P_{La}, P_{Pa}))_k$, $\Delta 4_k = (\max(P_{La}, P_{Pa}) - \widetilde{P}_{xy\frac{1}{2}})_k$. The simulation results are presented in Tab. 1 and Tab. 2.

[4] Other cases will be presented in the extended version of the paper.

Statistics	$x = 0.05$ $y = 0.05$	$x = 0.25$ $y = 0.25$	$x = 0.5$ $y = 0.5$	$x = 0.75$ $y = 0.75$	$x = 0.95$ $y = 0.05$	$x = 0.95$ $y = 0.95$	$x = 0.05$ $y = 0.05$
Mean (P_{xy}) (Std. Dev. $_{P_{xy}}$)	1.101 (0.038)	1.097 (0.035)	1.102 (0.038)	1.087 (0.030)	1.086 (0.031)	1.103 (0.037)	1.101 (0.038)
Mean ($\widetilde{P}_{xy0.5}$) (Std. Dev. $_{\widetilde{P}_{xy0.5}}$)	1.103 (0.038)	1.097 (0.034)	1.102 (0.038)	1.089 (0.031)	1.086 (0.031)	1.104 (0.038)	1.103 (0.038)
$card\{k : \Delta_{1k} < 0\}$	28	26	17	21	15	16	28
$card\{k : \Delta_{2k} < 0\}$	20	19	22	23	12	30	20
$card\{k : \Delta_{3k} < 0\}$	7	25	17	15	15	2	7
$card\{k : \Delta_{4k} < 0\}$	4	14	22	17	9	4	4

Table 1. Verifying the Laspeyres- Paasche bounding test for P_{xy} and $\widetilde{P}_{xy0.5}$ - Case 1 ($v^s = v^t = 0.05$; $u^s = u^t = 0.05$).

Statistics	$x = 0.05$ $y = 0.05$	$x = 0.25$ $y = 0.25$	$x = 0.5$ $y = 0.5$	$x = 0.75$ $y = 0.75$	$x = 0.95$ $y = 0.05$	$x = 0.95$ $y = 0.95$	$x = 0.05$ $y = 0.05$
Mean (P_{xy}) (Std. Dev. $_{P_{xy}}$)	1.066 (0.127)	1.143 (0.140)	1.103 (0.126)	1.043 (0.130)	1.067 (0.134)	1.143 (0.134)	1.066 (0.127)
Mean ($\widetilde{P}_{xy0.5}$) (Std. Dev. $_{\widetilde{P}_{xy0.5}}$)	1.082 (0.124)	1.148 (0.141)	1.103 (0.126)	1.023 (0.137)	1.067 (0.134)	1.131 (0.132)	1.082 (0.124)
$card\{k : \Delta_{1k} < 0\}$	28	28	39	44	21	34	28
$card\{k : \Delta_{2k} < 0\}$	22	25	32	30	33	31	22
$card\{k : \Delta_{3k} < 0\}$	3	26	39	34	19	3	3
$card\{k : \Delta_{4k} < 0\}$	7	20	32	21	27	5	7

Table 2. Verifying the Laspeyres- Paasche bounding test for P_{xy} and $\widetilde{P}_{xy0.5}$ - Case 2 ($v^s = v^t = 0.2$; $u^s = u^t = 0.2$).

Conclusions

The proposed and wide class of price indices ($\widetilde{P}_{xyz}$) has similar axiomatic properties as the geo-logarithmic price index family and in particular, each index from this family satisfies the Martini's minimal requirements. Moreover, the "ideal", superlative[5] Fisher price index belongs to the proposed family. In the simulation study we observe that indices P_{xy} and $\widetilde{P}_{xy0.5}$ may differ strongly for $x \neq 0.5$ or $y = 0.5$, i.e. we observe some differences[6] between expected index values (arithmetic means) calculated for their generated values. The precision of estimation of P_{xy} and $\widetilde{P}_{xy0.5}$ indices, i.e. the standard deviations of their generated values, are comparable with respect to size and they don't seem to depend on x and y. This is a practical conclusion: even if fluctuations of prices and quantities are large we observe similar volatility among price indices[7] from the same general class of indices. Finally, the most crucial difference between compared general class of indices is that the *probability*[8] of satisfying the Laspeyres-Paasche bounding test is bigger in the case of $\widetilde{P}_{xy0.5}$ index (it is much bigger for small (near 0) and big (near 1) values of x and y. In other words, we observe relatively fewer cases when the value of $\widetilde{P}_{xy0.5}$ index is outside of the interval determined by the Laspeyres and Paasche price indexes in comparison to analogical cases for P_{xy} formula (see Tables 1-2).

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[5] The term "superlative" comes from Diewert (1976). For more details see also Hill (2006).

[6] As a rule the difference between the above-mentioned indices is no more than 1 p.p.

[7] Obviously, in the simulation study we treat price indices as some random variables.

[8] The above-mention *probability* is estimated as a ratio of the number of generated cases when the considered price index fulfills the Laspeyres-Paasche bounding test and the total number of repetitions.

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