Spectral VaR test statistical properties

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Abstract

Developed primarily to describe physical phenomena like the vibrating string or heat flow, the spectral theory offers powerful tools applicable also in economics as means of statistical signal processing. The power spectrum, obtained from the time series as a Fourier transform of the autocovariance function, gives detailed information about the time structure of the process. The information content of the power spectrum is the same as the autocovariance, however these functions expose different aspects of the correlation structure of the time series. It has been shown that the spectral analysis may reveal hidden periodicities in the studied data. Thus for practical purposes the spectrum might be the more useful parameter than the autocovariance function itself.

Since the key issue in VaR model evaluation is to check for time dependence in the residual VaR failure series, the backtesting procedure may be based on the power spectrum instead of the autocovariance function. The study investigated statistical properties of the spectral VaR tests. The test evaluation was based on the Monte Carlo experiments designed in such a way that they reflect the volatility clustering phenomenon. The study showed that spectral VaR tests outperform the commonly used Markov test in detecting incorrect risk models.

Keywords: *spectral test, VaR test, martingale difference hypothesis*

JEL Classification: C22, C52, D53

1. Introduction

The development of the spectral theory was primarily motivated by studies of physical phenomena like the vibrating string or heat flow[2]. The underlying idea was to transform the observed signal from the time domain to the frequency domain to obtain information about the periodical structure of the process. The basic tool of signal processing in the spectral theory is the Fourier transform, by which the harmonic functions of sine and cosine are multiplied by the signal output and the resulting integration returns frequency information about the underling physical process. Through the decomposition into a linear combination of sines and cosines, it is estimated, from a finite record of a stationary data sequence, how the total power of the process is distributed over frequency. The transform does not generate any information loss, which means that the initial autocorrelation function can be reconstructed by the inverse Fourier transform.

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[2] An in-depth introduction to the Fourier analysis can be found in Stein and Shakarchi (2003).

The Fourier transform processes the function, changing its domain, in a way that exposes periodical properties. Therefore, although there is a one-to-one correspondence in the information content between the Fourier pair consisting of the explicit autocorrelation function and the power spectrum, they differ in a way the information is displayed[3]. Thus the power spectrum may be the more useful parameter than the autocorrelation function in practical applications. Especially it may constitute an attractive base for statistical test construction, offering possible power gains.

In VaR model evaluation the key issue, which determines specific backtesting procedures, is to check for independence in the residual series. It is required from a VaR-based risk model that the information contained in the base process is fully used, hence the residuals, which form a residual VaR failure series, do not exhibit any form of time dependence. The standard approach to capture time series dependence in subsequent VaR failures concentrates on testing the Markov property (Christoffersen, 1998). The extensive toolkit of VaR evaluation[4] involves also estimating sample autocorrelations (e.g. in a well-known Ljung Box test), regressing VaR violations on their lagged values (Engle and Manganelli, 2004) or checking unpredictability of the durations between failures (Christoffersen and Pelletier, 2004). Another method for testing dependence in time series is to examine the shape of the spectral density function (Berkowitz et al., 2011). The spectral VaR test utilize the property that in the case of a white noise the series has a particularly simple representation of a flat line in a frequency domain.

The aim of the study was to assess the performance of spectral methods in VaR model evaluation. Spectral VaR tests were examined through their basic statistical properties – the size and the power. The study involved test assessment in relation to the standard Markov-chain-based procedure and the comparative analysis of various testing statistics proposed within the spectral test framework. The statistical properties were investigated through the Monte Carlo method with simulation experiments based on the GARCH process, which guaranteed representation of the volatility clustering phenomenon.

The second section of the paper introduces spectral-based methods of testing independence in statistical time series and presents possible testing statistics. In the third section we report the

[3] A in-depth introduction to the spectral theory can be found in Koopmans (1995) and broad discussion on its application to statistical signal processing is presented in Stoica and Moses (2005). In Polish literature the application of spectral methods in econometrics was studied by Talaga and Zieliński (1986).

[4] For an overview of VaR theory see e.g. Christoffersen (2012). The comparative study of the basic VaR tests can be found in Małecka (2013).

results of a detailed Monte Carlo study and discuss spectral VaR test performance in terms of their size and power in finite sample setting. The final section summarizes and concludes.

2. Spectral-density-based VaR tests

Let us define the VaR failure process:

$$I_t = 1_{\{-x_t > VaR_p(x_t)\}} \tag{1}$$

where x_t represents the output of the statistical signal, interpreted usually as the return from investment[5] at time t and $VaR_p(x_t)$ is the value at risk of x_t at time t, $t = 1,...,T$, on the level of tolerance p, $p \in (0,1)$. Let us consider the hypothesis that the process $(I_t)_{t=0}^T$ is the martingale difference sequence $H_0 : E(I_t \mid \Omega_{t-1}) = p$, where Ω_t is the information set available at time t and $p \in (0,1)$ is the fixed tolerance level.

The Markov test statistic, used here as a benchmark, is formulated in terms of conditional probabilities of a single-step transition:

$$LR_{ind} = -2\log \frac{\hat{\pi}_1^{t_1}(1-\hat{\pi}_1)^{t_0}}{\hat{\pi}_{01}^{t_{01}}(1-\hat{\pi}_{01})^{t_{00}}\hat{\pi}_{11}^{t_{11}}(1-\hat{\pi}_{11})^{t_{10}}} \sim {}_{as}\chi^2_{(1)}, \tag{2}$$

$\hat{\pi}_1 = \frac{t_1}{t_0 + t_1}$, $t_0 -$ number of non-exceptions ($I_t = 0$), $t_1 -$ number of exceptions ($I_t = 1$), $\hat{\pi}_{01} = \frac{t_{01}}{t_0}$, $\hat{\pi}_{11} = \frac{t_{11}}{t_1}$, $t_{ij} -$ number of transitions form the state i to the state j, where state 0 is a non-exception and state 1 is an exception.

The spectral test refer to the properties of the autocorrelation and autocovariance function. The null in the above form implies that the autocorrelation function $\rho_I(k)$ and the autocovariance function $\sigma_I(k)$ of the random variable I_t equal zero for all lag orders k, $k \in Z$ (Berkowitz et al., 2011). The spectral test works on the idea of comparing the spectral density function f_I, which is the Fourier transform of the autocovariance function $\sigma_I(k)$:

$$f_I(\omega) = \frac{1}{2\Pi}\sum_{k=-\infty}^{\infty}\sigma_I(k)e^{-ik\omega}, \tag{3}$$

to the theoretical spectral density of the martingale difference process, which, under H_0, has a particularly simple shape of a flat function:

[5] Various possibilities to construct the rate of return and average return in both discrete and continuous time are discussed in Białek (2013).

$$f_I(\omega)=\frac{\sigma_I(0)}{2\Pi}=\frac{p(1-p)}{2\Pi}. \tag{4}$$

The periodogram estimate of the spectral density is given by:

$$P_T(\omega)=\frac{1}{2\Pi}\sum_{k=-(T-1)}^{T-1}\hat{\sigma}_I(k)e^{-ik\omega}. \tag{5}$$

The distance between the estimated spectral density and the theoretical flat line is measured by the expression:

$$\gamma_T(\omega)=\frac{1}{2\Pi}\left(\sum_{k=-(T-1)}^{T-1}\hat{\sigma}_I(k)e^{-ik\omega}-\sigma_I(0)\right). \tag{6}$$

The above formula represents a random function, hence it cannot be used directly as a test statistic. This function does not exhibit point convergence. Under H_0 the cumulated function converges to zero, which allows for measuring the discrepancy[6] between the observed series and the H_0 by the function:

$$\Gamma_T(\lambda)=\int_0^{\lambda}\left(P_T(\omega)-\frac{\hat{\sigma}_I(0)}{2\pi}\right)d\omega,\quad \lambda\in[0,\Pi]. \tag{7}$$

The test statistic is based on the modification of the formula (7):

$$\begin{aligned}U_T(t)&=\sqrt{2T}\int_0^{\Pi t}\left(\frac{P_T(\omega)}{\hat{\sigma}_I(0)}-\frac{1}{2\pi}\right)d\omega=\\&=\frac{\sqrt{2}}{\Pi}\sum_{h=1}^{T-1}\sqrt{T}\,\hat{\rho}_I(h)\frac{\sin h\Pi t}{h},\quad t\in[0,1],\end{aligned} \tag{8}$$

in which λ is converted to $\Pi t,\ t\in[0,1]$, the multiplication by $\sqrt{2}$ is used to facilitate computation of the asymptotic distribution[7] and the normalization factor $\frac{\sigma_I(0)}{\sqrt{T}}$ is introduced. That gives the formula being the function of the sample autocorrelation normalized by $\sqrt{T}$. In opposite to the expression (7), the modified statistic is robust against heteroskedasticity (Durlauf, 1991).

Under H_0 it holds that $U_T(t)\xrightarrow{d}B(t),\ t\in[0,1]$ where $B(t)$ is the Brownian bridge on $[0,1]$[8]. Using the above convergence the martingale property test can be conducted through a number of test statistics, which map the random function $U_T(t)$ into the random variable. The

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[6] More about measures of discrepancy and their statistical properties can be found in Żądło (2013).
[7] The proof can be found in Durlauf (1991).
[8] The Brownian bridge is the Wiener process conditional on $W(1)=0,$ therefore $B(t)=W(t)\,|\,\{W(1)=0\},\ t\in[0,1].$

following statistics are proposed: Anderson-Darling statistic $SD_{AD} = \int_0^1 \frac{U(t)^2}{t(1-t)}$, Cramer von Mises statistic $SD_{CVM} = \int_0^1 U(t)^2$, Kolmogorov-Smirnov statistic $SD_{KS} = \sup_{t\in[0,1]} |U(t)|$ and Kuiper statistic $SD_{Kui} = \sup_{0\le s,t\le 1} |U(t)-U(s)|$ (Durlauf, 1991).

3. Simulation study

Statistical properties of test statistics based on spectral density function and the LR_{ind} statistic of the Christoffersen's Markov test, used as a benchmark VaR test, were compared through the simulation study. The size and the power were estimated as the proportion of rejections under the null (type-one errors) and the proportion of rejections under the alternative respectively. The size assessment was done through generating i.i.d. Bernoulli samples with the probability $p = 0.05$, equal to the chosen VaR tolerance level.

Test	**Series length**			
	250	**500**	**750**	**1000**
LR_{ind}	0.071	0.083	0.122	0.137
SD_{KS}	0.028	0.029	0.033	0.032
SD_{Kui}	0.028	0.031	0.033	0.035
SD_{CVM}	0.040	0.039	0.042	0.043
SD_{AD}	0.046	0.047	0.047	0.049

Table 1. Size estimates of the spectral VaR tests.

For the power comparison we adopted the Monte Carlo simulation technique, where we replaced the theoretical distribution of the test statistics by their sample analogues simulated under the null (Dufour, 2006). The test performance was assessed through the experiment with conditionally heteroscedastic return data obtained from the GARCH model[9], which reflects practical market conditions characterized by the volatility clustering phenomenon. VaR

[9] The power evaluation experiment used the GARCH-normal model with variance equation of the form $h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}$ with parameter values $\omega = 0.000001$, $\beta = 0.85$ and α parameter value on a relevant level to ensure the required value of ρ (Drachal, 2015; Fiszeder, 2009; Małecka, 2011).

forecasts were set to the level of the 0.05 quantile of the unconditional distribution of the return process. The strength of the correlation in VaR failure series was assessed by the correlation coefficient of the squared returns ρ, whose value was set to 0.1, 0.3 and 0.5 in subsequent variants of the simulation experiment. The size and power estimates were computed over 10,000 replications for sample sizes $T = 250, 500, 750, 1000$ with the level of significance set to 5%.

Rejection rates obtained under the null for all considered test statistics – LR_{ind}, SD_{KS}, SD_{Kui}, SD_{CVM}, SD_{AD} – showed that the empirical size of the Markov LR_{ind} test was much further from the nominal test size than the size results obtained for spectral tests (Table 1). Moreover the null distribution of the LR_{ind} test statistic did not show convergence to the theoretical distribution, which translated into test size of over 10% for series of 750 observations or more. Empirical size for spectral tests was closer to the nominal 5% level, with best results fitting into the interval between 4.5% and 5% independent of the sample size.

In the group of spectral tests the estimated test size was generally below the nominal 5%, which indicated the conservative character of the tests. The largest compliance between the empirical and asymptotic distribution was observed in the case of the SD_{AD} test, based on Anderson-Darling statistic.

The results of the power comparison based on the GARCH experiment did not show clear superiority of any of the approaches. Spectral tests exhibited more power in detecting low-scale correlation ($\rho = 0.1$) in time series, while in the case of the evident correlation ($\rho = 0.3$ and $\rho = 0.5$) they were outperformed by the LR_{ind} test statistic.

The study showed fast growth in the test power with lengthening the time series, especially with shift from 250 to 500 observations. For 500 observations and correlation characterized by $\rho = 0,3$ or $\rho = 0,5$ the estimated power exceeded 40% in the case of all tests, while for 1,000 observations it got over 70% in most cases. The comparative analysis of the power estimates of the four considered spectral test statistics showed the superiority of the SD_{AD} test, which also outperformed other tests in the size exercise.

Test	ρ	**Series length**			
		250	**500**	**750**	**1000**
	0.1	0.10	0.13	0.17	0.19
LR_{ind}	0.3	0.36	0.59	0.76	0.86
	0.5	0.64	0.89	0.97	0.99

SD_{KS}	0.1	0.12	0.20	0.25	0.28
	0.3	0.23	0.46	0.59	0.76
	0.5	0.23	0.54	0.71	0.84
SD_{Kui}	0.1	0.11	0.17	0.21	0.23
	0.3	0.22	0.43	0.57	0.69
	0.5	0.23	0.51	0.72	0.84
SD_{CVM}	0.1	0.13	0.18	0.21	0.25
	0.3	0.25	0.43	0.55	0.71
	0.5	0.26	0.53	0.68	0.82
SD_{AD}	0.1	0.13	0.25	0.30	0.34
	0.3	0.28	0.58	0.72	0.84
	0.5	0.30	0.67	0.81	0.91

Table 2. Power estimates of the spectral VaR tests.

4. Conclusion

The study explored application of the spectral theory to risk analysis based on the VaR model. Spectral-based approach was adopted in risk model evaluation to test for correlation in the VaR failure series. The paper presented principles of the spectral test construction which utilized the flat shape of the theoretical spectral density of the white noise process.

Assessment of test properties through the Monte Carlo method showed spectral-based approach superiority in terms of the test size. The empirical size of the standard Markov-based VaR test was much further from the nominal test size than the size results obtained for spectral tests. The power exercise indicated that spectral tests are superior in detecting low-scale correlation in time series, while in the case of the evident correlation they were outperformed by the Markov-based test statistic. Among spectral tests the Anderson-Darling statistic outperformed other statistics both in terms of the size and the power.

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