The use of asymmetric loss function for estimating premium rates in motor insurance

Anna Szymańska[1]

Abstract: In motor insurance assigned premium is calculated as the product of the basic premium and premium rates. Contribution rate can be estimated using Bayesian methods. In the paper Bayesian estimators of premium rates with LINEX loss function and exponential function were determined. It was evaluated how the choice of loss function affects discounts and increases in premiums and tariffication efficiency of bonus-malus system.

Keywords: *bayes estimators, exponential loss function, LINEX loss function, premium rates*

JEL Classification: C11, C440, G22
AMS Classification: 60J20, 62P05

1. Introduction

Insurance market in Poland is growing rapidly as evidenced by the continuous changes of individual insurers' ownership in the market. At the same time in car insurance liability since 2008 support loss appears. Insurance companies must measure risk in their tariffs more effectively. In car insurance CR tariffication is a two-step. In the first stage - called a piori - base premium is determined on the basis of known risk factors (in Poland it is usually the vehicle registration region and engine displacement). Then in the base premium increases and discounts are taken into account, mainly resulting from the claims experience in the previous insurance period. This stage is called a posteriori tariffication. The rules for granting increases and discounts are determined individually by the insurance company and are called bonus-malus system. Increases and discounts are determined as a percentage of the base premium and are called net premium rates. In the paper an application of Bayesian methods to estimate the rates of net premiums for liability insurance is presented. In the construction of estimators the exponential loss function and linear-exponential function are used. Net premium is determined on the basis of the expected value principle. The aim of the paper was to assess whether the form of the loss function significant affects the value of the premium rates and what influence on the results of the estimation has the choice of constants in loss function. Moreover the tariffication effectiveness of bonus-malus systems constructed on the basis of the estimated rates of premium was assessed.

[1] University of Lodz, Department of Statistical Methods, Rewolucji 1905 r. 41/43, 90-214 Lodz, Poland, szymanska@uni.lodz.pl

2. Bayesian estimators of the parameter *θ* for different loss functions

Let $(K_1,...,K_n)$ be a simple random sample, and *K* is a random variable whose distribution depends on the parameter *θ* assuming values from set Θ. Denote by *D* the set of all possible decisions, and by *d* an element of this set being a one-dimensional random variable. Let (Θ, D, L) be a statistical game with set of states Θ, the set of all possible decisions *D* and the loss function *L*. The loss function we call a function defined on the set $\Theta \times D$, whose values are the size of the losses incurred as a result of taking certain decision *d* reffering to the value of parameter *θ*. Consider the following loss functions: exponential and linear-exponential (LINEX) [7], [4], given, respectively by the formulas:

$$L^{(I)}(\theta, d) = \exp[-a(\theta - d)],\ a > 0, \tag{1}$$

$$L^{(II)}(\theta, d) = b[\exp(a(\theta - d)) - a(\theta - d) - 1],\ a \neq 0, b > 0. \tag{2}$$

Typically in the Bayesian estimation, the quadratic loss function is used. The quadratic loss function means that in the case of overestimation or underestimation of the unknown parameter *θ* by the same amount has the same meaning for the decision maker. But in the case of insurance it does not always have to be like this. Overestimation of premiums may cause loss of customer, and underestimation of premiums may cause losses for the insurance company. Zellner was the first who proposed a non-symmetrical loss function called linear-exponential [10]. In the case of the LINEX loss function parameter *b* is the scale parameter and one usually assumes that *b* = 1, the parameter *a* is a shape parameter and its sign reflects the direction of the asymmetry. The higher the $|a|$ is, the greater is the asymmetry of LINEX function. Let K be a random variable representing the number of claims in a given year for a single policy. Then the vector $(k_1, k_2, \ldots, k_t)$ will be a vector of an observed for t years the number of claims for a given policy. Claims parameter θ_{t+1} in year *t*+1 for the policy described observation vector $(k_1, k_2, \ldots, k_t)$ is unknown. This parameter can be estimated with using the Bayesian estimator $\hat{\theta} = \lambda_{t+1}$ based on the observation of vector $(k_1, k_2, \ldots, k_t)$. Suppose that the distribution of the number of claims in the motor insurance is the Poisson distribution [9] with a parameter θ

$$P(K = k) = \exp(-\theta)\frac{\theta^k}{k!},\ k = 0,1,... \tag{3}$$

Let the parameter θ be a random variable with the priori gamma distribution with parameters α and β of the density function given by the equation

$$\pi(\theta)=\frac{\beta^{\alpha}\theta^{\alpha-1}}{\Gamma(\alpha)}\exp(-\beta\theta) \quad ,\theta>0,\alpha>0,\beta>0 \,. \tag{4}$$

Then the distribution of the number of losses in the portfolio is negative binomial. Bayesian estimators of parameter for the loss function described by equations (1) - (2) have respectively the forms:

$$\hat{\theta}_B^{(I)}=\frac{\alpha}{\beta}\left[1-\frac{t}{a}\ln\left(1+\frac{a}{\beta+t}\right)\right]+\frac{k}{a}\frac{\beta}{\alpha}\ln\left(1+\frac{a}{\beta+t}\right) \tag{5}$$

$$\hat{\theta}_B^{(II)}=\frac{1}{a}(\alpha+k)\ln\frac{\beta+t}{\beta+t-a}, \tag{6}$$

where $\beta+t-a>0$ [5], [8], [1].

3. Estimators of net premiums rates

In motor insurance individual net premium in period t+1 is determined from the equation [5]:

$$\Pi(X,K)=(EX)\cdot(EK)\cdot b_{t+1}\,. \tag{7}$$

Where $\Pi(X,K)$ - is individual net premium in period t+1, EX - the expected value of a single damage EK - the expected value of the number of claims from given insurance policy, b_{t+1}-premium rate in period $t+1$. Each insurance company has to specify the percentage values b_{t+1} - called premium rates. Assume that $EX=1$ and that the distribution of the number of losses in the portfolio is negative binomial, equation (7) has the form:

$$\Pi(K)=\frac{\alpha}{\beta}\cdot b_{t+1}. \tag{8}$$

Assume that the net premium is calculated on the basis of the principle of the expected value given by the formula:

$$\Pi(K)=(1+Q)EK, \tag{9}$$

where $Q \geq 0$ is called safety factor, which is determined by the insurer. Taking into account the assumptions about the form of distribution of the number of claims and using Bayesian estimators given in formulas (5) - (6), we get the following form of net premium rates, which should be paid by the insured, who after t years reported k damages:

$$b_{t+1}^{(I)}(K) = (1+Q)\left[1 - \frac{t}{a}\ln\left(1+\frac{a}{\beta+t}\right) + \frac{k}{a}\frac{\beta}{\alpha}\ln\left(1+\frac{a}{\beta+t}\right)\right]\cdot 100\% \qquad (10)$$

$$b_{t+1}^{(II)} = (1+Q)\frac{\beta}{a\alpha}(\alpha+k)\ln\frac{\beta+t}{\beta+t-a}\cdot 100\%. \qquad (11)$$

4. Selected measures of effectiveness tariffication of bonus-malus systems

Let the process of the moves of insured between classes be a finite homogeneous Markov chain with transition probability matrix **M** and the transition probability $p_{ij}(\lambda)$ of the insured with damages λ intensity ratio from bonus-malus C_i class to C_j class in one year [2].

The most common measure assessing the bonus-malus system is the overall efficiency $\eta(\lambda)$ (called elasticity of the mean stationary premium with respect to the claim frequency), called the efficiency of the system by Loimaranta [5], [6] and defined by the formula:

$$\eta(\lambda) = \frac{B'(\lambda)}{B(\lambda)}\cdot\lambda = \frac{dB(\lambda)}{B(\lambda)}\bigg/\frac{d\lambda}{\lambda} \qquad (12)$$

where the average asymptotic premium for a single period after reaching steady-state by the system is:

$$B(\lambda) = \sum_{i=1}^{s} a_j(\lambda)\cdot b_j = \bar{a}(\lambda)\bar{b} \qquad (13)$$

and b_j - premium in the class j, $\mathbf{b}=(b_1, \ldots, b_s)$ is a vector of premiums and $\mathbf{a}(\lambda)=[a_1(\lambda),\ldots a_j(\lambda)]$ is a vector whose components are the probabilities of belonging insured to the j-th class after the system reaches a steady state, wherein $\sum_{j=1}^{s} a_j(\lambda) = 1$. The overall effectiveness is a flexibility of average premium $B(\lambda)$ relative to the level of risk λ and allows to assess to what extent drivers are judged by the system. In perfect condition $B(\lambda)$ should be an increasing function of λ such that $\eta(\lambda)=1$. The effectiveness defined in this way has two drawbacks: first, steady-state process can not be achieved, and second the effectiveness estimates all the drivers together. The effectiveness of the system can also be measured by the average asymptotic premium, defined by (13). The insurance company with the highest average stationary premium reaches best technical result of the insurance activity. Average asymptotic premium of less than 100% indicates a high concentration of policies in classes with discounts. Another measure of tariffication efficiency is the relative steady expected level of premium (called *relative stationary average level*), denoted by RSAL and described the relationship:

$$RSAL(\lambda) = \frac{B(\lambda) - \min_j(b_j)}{\max_j(b_j) - \min_j(b_j)}. \qquad (14)$$

It is difficult to specify an optimum value of the indicator RSAL. According to the author of this measure in an ideal system, this indicator should be 0.5 for the average frequency of damage [5]. Small values of RSAL mean that the system will be in imbalance, and with the passage of time, most of policies will be in classes with the highest discounts. High indicator values mean that the distribution of policies among the classes BMS is uniform. However, this measure is criticized because of the impact of the maximum rate on the value of the indicator.

5. Empirical examination

For the study the following parameters of the negative-binomial distribution are taken: $\alpha = 1{,}6131; \beta = 16{,}1384$ determined on the basis of the data on the damages of one of the Belgian insurance companies, presented in paper [5]. In figures (1) and (2) the premium rates estimated for the exponential loss function, and linearly-exponential with various shape parameters a are compared. In the literature, the parameter of LINEX loss function shape is often associated with a measure of Arrow-Pratto risk aversion and bounded within to the limits $0{,}4 \le a \le 11{,}5$. However, in the actuarial literature the studies considering different values of this parameter, including negative [1], [3] can be found. In case of real data, limits are given by $\beta + t - a > 0$ which is domain of the logarithm in formula (11), so in practice the value of the parameter $|a| < \tilde{\beta}$.

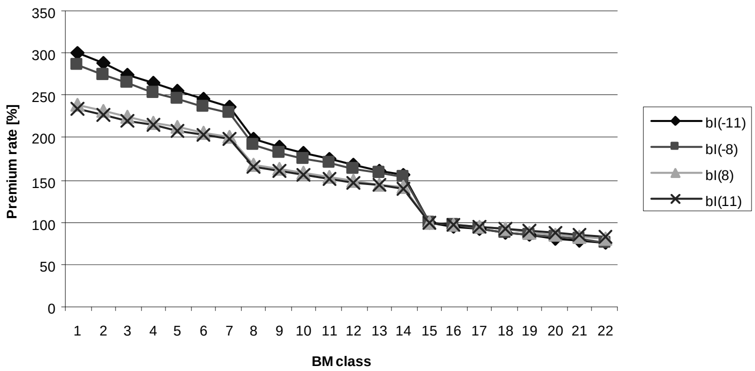

Fig. 1. The premium rates estimated on the basis of of the estimator *bI* -formula (10) (with the parameters *a* equal respectively -11, -8, 8, and 11).

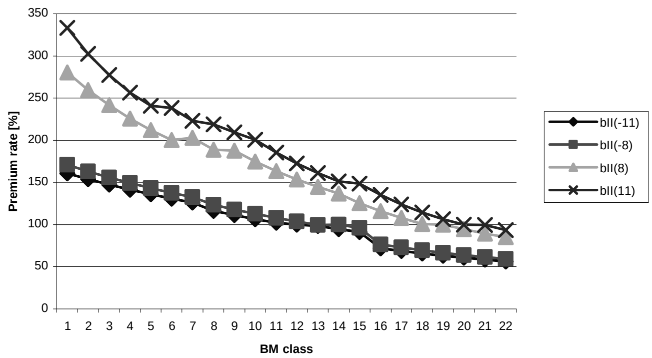

Fig. 2. The premium rates estimated on the basis of the estimator *bII* - formula (11) (with the parameters *a* equal respectively -11, -8, 8, and 11).

In case of the exponential loss function, for the smallest values of the parameter $a<0$, rates are highest in upward classes, smallest in discount classes. The positive parameter gives the lowest rates in upward classes, the largest in discount classes. However, the differences are small in this case. The highest values of rates were obtained for LINEX function with the positive, highest shape parameter $a=11$. For the values of the parameter $a>0$ the overestimation of premium rates will cause much higher losses than underestimation. For $a<0$ the opposite. The lowest rates were obtained with LINEX loss function with parameter $a=-11$. Table (1) shows the proposed bonus-malus system, where b1 to b22 are premium rates estimated by using the previously constructed estimators. For further research eight, bonus-malus-systems with the same rules of movement between classes and different rates of premiums in the classroom BMS are received. So constructed, the bonus-malus systems were evaluated using efficiency measures described by formulas (12) - (14). The results are presented in figure (3). The flexibility of studied bonus-malus system is not significantly different. However, the stationary premium is most favorable to the estimators received with the use of loss function LINEX with the largest shape parameter. The lowest values of stationary premium were obtained using the LINEX loss function with the smallest shape parameter. The RSAL measurement values are similar for all functions of loss and is approximately 0.03.

BM class	Premium rate [%]	Number of damages in year		
		0	1	2 and more
1	b_1	2	1	1
2	b_2	3	1	1
3	b_3	4	2	1
4	b_4	5	3	1
5	b_5	6	4	1
6	b_6	7	5	1
7	b_7	8	6	1
8	b_8	9	7	1
9	b_9	10	8	1
10	b_{10}	11	9	1
11	b_{11}	12	10	1
12	b_{12}	13	11	1
13	b_{13}	14	12	1
14	b_{14}	15	13	1
15	b_{15}	16	14	1
16	b_{16}	17	15	2
17	b_{17}	18	16	3
18	b_{18}	19	17	4
19	b_{19}	20	18	5
20	b_{20}	21	19	6
21	b_{21}	22	20	7
22	b_{22}	22	21	8

Table 1 Bonus-malus system.

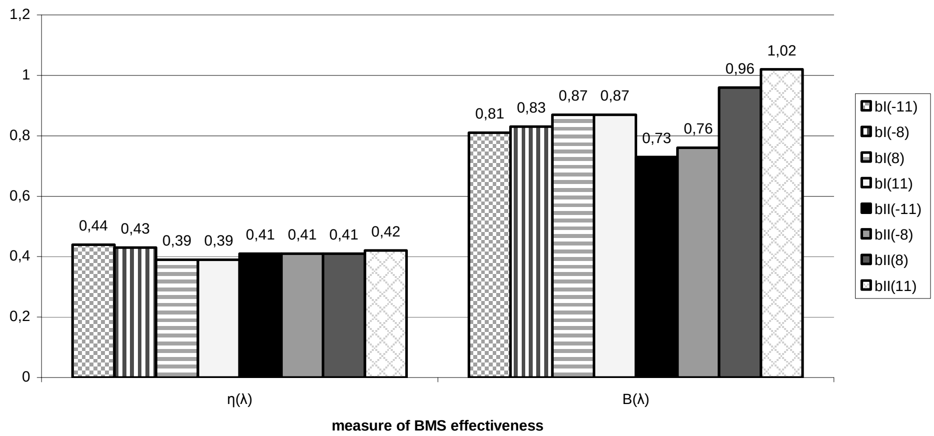

Fig. 3. Selected measures of tariffication effectiveness of bonus-malus system ($\lambda = 0.1$).

6. Conclusion

In ideal bonus-malus systems, the efficiency is 1. The examined systems have efficiency of order about 0.4. This means that the increase of the intensity of the damage by 1% results in increase of the average asymptotic premium only by 0.4%. The higher is the stationary premium of systems the better it is. It can be concluded that the choice of loss function significantly affect the value of stationary premium. RSAL measurement values are very low and indicate a concentration of policies in discount classes. They are not significantly different in case of different loss functions. Considering the choice of loss functions for estimation of premium rates from the insurer's point of the view, the LINEX loss function with the positive, fairly big shape parameter seems to be justified. The losses due to the overestimation of rates should not be so noticeable for the insurer especially that usually in discount classes there is a small percentage of policies.

References

[1] Boratyńska, A., 2008. Posterior Regret Gamma-Minimax Estimation of Insurance Premium in Collective Risk Model. ASTIN Bulletin 38, 277-291.

[2] Bonsdorff, H., 1992. On the convergence rate of Bonus-Malus systems. ASTIN Bulletin 22, 217-223.

[3] Jasiulewicz, H., Kordecki, W., 2011. Credibility premiums using asymmetric loss functions. Research Papers of Wrocław University of Economics 207, 101-117.

[4] Jokiel-Rokita, A., 2011. Bayes sequential estimation for a particular exponential family of distributions under LINEX loss. Metrika 74 (2), 211-219.

[5] Lemaire, J., 1985. Automobil Insurance Actuarial Models. Boston: Kluwer Academic Publishers.

[6] Lomiranta, K., 1972. Some asymptotic properties of bonus-malus systems. ASTIN Bulletin 6, 233-245.

[7] Męczarski, M., Zieliński, R., 1997. Bayes optimal stopping of homogeneous Poisson process under LINEX loss function and variation in the prior. Applicationes Mathematicae 24 (4), 457-463.

[8] Niemiro, W., 2006. Bayesian prediction with an asymmetric criterion in a nonparametric model of insurance risk. Statistics 40, 353-363.

[9] Walhin, J.-F., Paris, J., 2000. The true claim amount and frequency distributions within a bonus-malus system. ASTIN Bulletin 30, 391-403.

[10] Zellner, A., 1986. Bayesian estimation and prediction using asymmetric loss functions. Journal of the American Statistical Association 81, 446-451.