Bayesian statistical tests for parameters of structural changes model with Bernoulli variable

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Abstract

In this paper a structural changes model with Bernoulli variables X_1, X_2 is considered. These variables appear with probabilities λ and $1-\lambda$, respectively. Hypotheses about parameters of Bernoulli distributions and parameter λ can be verified using Bayesian statistical tests. As a result of the use of the Bayesian statistical tests, the decision of the acceptance of the hypothesis for which the posterior risk is lower, is made. The risk depends on the prior parameter's distribution and the loss function. Bayesian statistical tests are constructed when the prior distributions of parameter λ and the parameters of the Bernoulli distributions are the uniform distributions on the interval $(a,b)\subset(0,1)$.

Keywords: *Bayesian test, structural changes model, Bernoulli distribution*

***JEL Classification*:** C11

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1. Introduction

The presented model of structural changes with two Bernoulli random variables X_1, X_2 is applied in different social, economical, biological and medical researches. These variables occur with the probability λ and $1-\lambda$, respectively, and one of these variables is always realized. The considered model characterized by three parameters: probability λ and probabilities of realization one of two variables $p_1,\ p_2$.

Hypotheses about models' parameters can be verified by means of the Bayesian tests. The problem with verification of such hypotheses occurs when we have information about the prior distribution of parameters (e.g. [2], [5], [8]). Using Bayesian tests we accept the hypothesis whose posterior risk is smaller. This risk depends on the prior parameter's distribution, on the loss function and on the sampling scheme. We assume that the prior distribution is the uniform distribution on the interval $(a,b)\subset(0,1)$ and we apply independent sampling scheme.

We analyze the properties of Bayesian tests in a simulation study.

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2. Idea of Bayesian tests

Let Y be a random variable and θ the investigated parameter of the distribution of this variable. Let $g(\theta)$ be the prior distribution of θ.

Bayesian statistical test. Bayesian tests are presented in [1], [4], [6], [9].

We consider the following hypotheses about the value of this parameter: $H_0 : \theta \le \theta_0$, $H_1 : \theta > \theta_0$ and verify its using the We accept the hypothesis whose the posterior risk is smaller. The posterior risk is defined for the following the loss function:

$$L(\theta, d_0) = \begin{cases} 0 & \text{dla } \theta \le \theta_0 \\ c_0 & \text{dla } \theta > \theta_0 \end{cases}, \qquad L(\theta, d_1) = \begin{cases} 0 & \text{dla } \theta > \theta_0 \\ c_1 & \text{dla } \theta \le \theta_0 \end{cases}, \tag{1}$$

where $\mathbf{y}$ is the realization of random sample $\boldsymbol{Y} = (Y_1, Y_2, ..., Y_n)$, d_0, d_1 are decisions about acceptance of hypothesis, respectively, H_0, H_1 and c_0, c_1 are fixed values. The risk function is defined as $r(d, \mathbf{x}) = EL(\theta, d(\mathbf{x}))$.

For continuous prior distribution of parameter θ, the posterior distribution has the form:

$$g(\theta|\mathbf{y}) = \frac{f(\mathbf{y}|\theta) \cdot g(\theta)}{\int_{\Theta} f(\mathbf{y}|\theta) \cdot g(\theta) d\theta} \tag{2}$$

where $f(\mathbf{y}|\theta)$ is the likelihood function of random sample $\boldsymbol{Y}$.

The acceptance of the null hypothesis is conditioned by the fulfillment of the inequality ([3]):

$$\frac{P(\theta \le \theta_0 | \mathbf{y})}{P(\theta > \theta_0 | \mathbf{y})} > \frac{c_0}{c_1} \tag{3}$$

or

$$P(\theta \le \theta_0 | \mathbf{y}) > \frac{c_0}{c_0 + c_1}, \tag{4}$$

hence

$$\int_{-\infty}^{\theta_0} g(\theta|\mathbf{y}) d\theta > \frac{c_0}{c_0 + c_1}. \tag{5}$$

In other cases we accept the alternative hypothesis.

3. Structural changes model with Bernoulli variable and its parameters verification

We consider a statistical model of structural changes with two random variables X_1, X_2 with the probability function, respectively, $x_1 = \begin{cases} 1 & \text{with probabil. } p_1 \\ 0 & \text{with probabil. } 1 - p_1 \end{cases}$ and $x_2 = \begin{cases} 1 & \text{withprobabil.} p_2 \\ 0 & \text{withprobabil} 1 - p_2 \end{cases}$, where $p_1, p_2 \in (0,1)$ and $p_1 \neq p_2$. These variables don't occur simultaneously. The probability that variable X_1 occurs is equal to λ and for variable X_2 the respective probability is equal to $1-\lambda$.

This model can be written as $y_t = \begin{cases} x_{1t} & \text{withprobabil} \lambda \\ x_{2t} & \text{withprobabil} 1 - \lambda \end{cases}$, where $t = 1, 2, ..., T$.

The variable Y_t has Bernoulli distribution with parameter $p = p_1\lambda + p_2(1-\lambda)$, because

$$P(y_t = 1) = P(y_t = 1 | x_{1t} = 1) \cdot P(y_t = x_{1t}) + P(y_t = 1 | x_{2t} = 1) \cdot P(y_t = x_{2t})$$

$$= p_1\lambda + p_2(1-\lambda).$$

The application of Bayesian methods to the estimation of parameters of this model is presented in [7]. Now, we consider Bayesian tests, which allow to verify hypothesis about models' parameter, if the prior distribution of the parameter verified is known.

Firstly, let us assume that the values of p_1, p_2 are known and parameter λ is unknown, but it has the prior uniform distribution on the interval (a,b):

$$g(\lambda) = \begin{cases} \frac{1}{b-a} & \text{dla } \lambda \in (a,b), \\ 0 & \text{dla } \lambda \notin (a,b), \end{cases} \tag{6}$$

where $0 \le a < b \le 1$.

We verify hypotheses about parameter λ. They have the following form:

$$H_0 : \lambda \le \lambda_0, \tag{7}$$

$$H_1 : \lambda > \lambda_0. \tag{8}$$

The likelihood function determined on the basis of the probability function of discrete random variable *Y*, is expressed by the following formula:

$$f(\mathbf{y}, p_1, p_2, \lambda) = (p_1\lambda + p_2(1-\lambda))^m (1 - p_1\lambda - p_2(1-\lambda))^{T-m}, \quad m = \sum_{i=1}^{T} y_i. \tag{9}$$

The posterior distribution of parameter λ has the form:

$$g(\lambda|\mathbf{y}) = \frac{(p_1\lambda + p_2(1-\lambda))^m (1 - p_1\lambda - p_2(1-\lambda))^{T-m}}{\int_a^b (p_1\lambda + p_2(1-\lambda))^m (1 - p_1\lambda - p_2(1-\lambda))^{T-m} d\lambda} = \frac{(p_1 - p_2)(p_1\lambda + p_2(1-\lambda))^m (1 - p_1\lambda - p_2(1-\lambda))^{T-m}}{B(m+1, T-m+1)(F(p_1 b + p_2(1-b)) - F(p_1 a + p_2(1-a)))}, \tag{10}$$

where F is the cdf of the beta distribution $B(m+1, T-m+1)$.

We calculate the value of probability:

$$P(\lambda \le \lambda_0|\mathbf{y}) = \int_a^{\lambda_0} g(\lambda|\mathbf{y}) d\lambda = \frac{(p_1 - p_2)\int_a^{\lambda_0} (p_1\lambda + p_2(1-\lambda))^m (1 - p_1\lambda - p_2(1-\lambda))^{T-m} d\lambda}{B(m+1, T-m+1)(F(p_1 b + p_2(1-b)) - F(p_1 a + p_2(1-a)))} = \frac{F(p_1\lambda_0 + p_2(1-\lambda_0)) - F(p_1 a + p_2(1-a))}{F(p_1 b + p_2(1-b)) - F(p_1 a + p_2(1-a))} \tag{11}$$

and compare it with $\frac{c_0}{c_0 + c_1}$ and accept either H_0 or H_1 (see formula (5)).

Secondly, we consider the case, when one of parameters p_i, $i = 1, 2$, is unknown and other parameters are known. Let us assume that p_1 is unknown and its prior distribution is the uniform distribution on the interval (a, b), where $0 \le a < b \le 1$. We verify the following hypotheses:

$$H_0 : p_1 \le p_0, \tag{12}$$

$$H_1 : p_1 > p_0. \tag{13}$$

In this case the posteriori distribution has the form:

$$g(p_1|\mathbf{y}) = \frac{(p_1\lambda + p_2(1-\lambda))^m (1 - p_1\lambda - p_2(1-\lambda))^{T-m}}{\int_a^b (p_1\lambda + p_2(1-\lambda))^m (1 - p_1\lambda - p_2(1-\lambda))^{T-m} dp_1} = \frac{\lambda(p_1\lambda + p_2(1-\lambda))^m (1 - p_1\lambda - p_2(1-\lambda))^{T-m}}{B(m+1, T-m+1)(F(b\lambda + p_2(1-\lambda)) - F(a\lambda + p_2(1-\lambda)))}. \tag{14}$$

We calculate the probability using the cdf of the Beta distribution:

$$P(p_1 \le p_0|\mathbf{y}) = \int_a^{p_0} g(p_1|\mathbf{y})d\lambda = \frac{\int_a^{p_0} (p_1\lambda + p_2(1-\lambda))^m (1-p_1\lambda - p_2(1-\lambda))^{T-m} dp_1}{\int_a^{b} (p_1\lambda + p_2(1-\lambda))^m (1-p_1\lambda - p_2(1-\lambda))^{T-m} dp_1} = \tag{15}$$

$$= \frac{F(p_0\lambda + p_2(1-\lambda)) - F(a\lambda + p_2(1-\lambda))}{F(b\lambda + p_2(1-\lambda)) - F(a\lambda + p_2(1-\lambda))}$$

and compare it with $\frac{c_0}{c_0 + c_1}$. Similarly, we can formulate and verify hypotheses about parameter p_2.

4. Monte Carlo analysis of the properties of the Bayesian tests

In order to analyze the properties of the Bayesian tests for parameters of structural changes model, we assume that the loss function is defined by the formula (1) and $c_0 = c_1$.

First, we consider structural model in which $p_1 = 0.1$, $p_2 = 0.6$ and λ is unknown but it has the uniform distribution on the interval $(0, 0.5)$. We generated the value λ and obtained 0.2438, then we drew a sample of 100 elements and verified the null hypothesis $H_0 : \lambda \le 0.3$ against $H_1 : \lambda > 0.3$. The number of “ones” was equal to 49. The plot of the posterior distribution is presented in the figure 1.

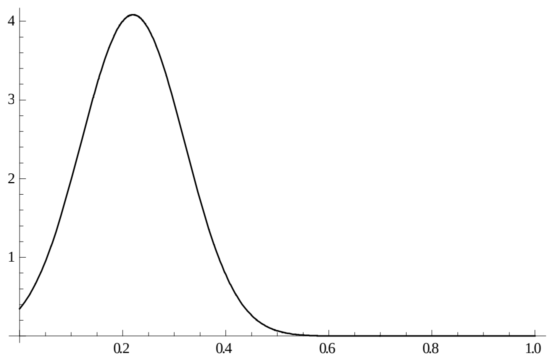

Fig. 1. The posterior distribution of λ.

The test statistics was equal to 0.790146 and we made a decision of the acceptance of hypothesis H_0. We made R=10000 repetitions to investigate the properties of the Bayesian test for hypotheses (7) and (8). We assume, that the uniform distribution on the interval

$(0, 0.5)$ is the prior distribution of λ The results of the Monte Carlo analysis for selected values of p_1, p_2, and λ_0 and different sample sizes are presented in tables 1 and 2.

λ_0	***n***	**Model parameters** $p_1 = 0.1$, $p_2 = 0.8$	$p_1 = 0.1$, $p_2 = 0.6$	$p_1 = 0.1$, $p_2 = 0.4$	$p_1 = 0.2$, $p_2 = 0.5$	$p_1 = 0.2$, $p_2 = 0.7$
0.2	100	0,0410	0.0596	0.1010	0.1032	0.0584
	200	0.0283	0.0490	0.0746	0.0819	0.0457
0.3	100	0.0326	0.0752	0.1097	0.1235	0.0742
	200	0.0235	0.0571	0.0821	0.0895	0.0428
0.4	100	0.0484	0.0796	0.1379	0.1342	0.0797
	200	0.0314	0.0535	0.0917	0.0951	0.0544

Table 1 The percentage of the acceptance of false H_0 for parameter λ.

λ_0	***n***	**Model parameters** $p_1 = 0.1$, $p_2 = 0.8$	$p_1 = 0.1$, $p_2 = 0.6$	$p_1 = 0.1$, $p_2 = 0.4$	$p_1 = 0.2$, $p_2 = 0.5$	$p_1 = 0.2$, $p_2 = 0.7$
0.2	100	0.0528	0.0870	0.1254	0.1295	0.0813
	200	0.0348	0.0566	0.0935	0.0954	0.0529
0.3	100	0.0537	0.0647	0.1086	0.1138	0.0656
	200	0.0321	0.0467	0.0773	0.0809	0.0529
0.4	100	0.0390	0.0416	0.0535	0.0407	0.0435
	200	0.0249	0.0436	0.0482	0.0553	0.0461

Table 2 The percentage of the acceptance of false H_1 for parameter λ.

Next, we consider a structural model in which $\lambda = 0.2$, $p_2 = 0.8$ and p_1 is unknown but it has the uniform distribution on the interval $(0, 0.5)$. We generate the value p_1 and obtain 0.09965, then we draw a sample of 100 elements and verify hypothesis $H_0 : p_1 \le 0.3$ against $H_1 : p_1 > 0.3$. The number of "ones" is equal 56. The plot of the posterior distribution is presented on the figure 2. The test statistics is equal to 0.971325 and we accept H_0.

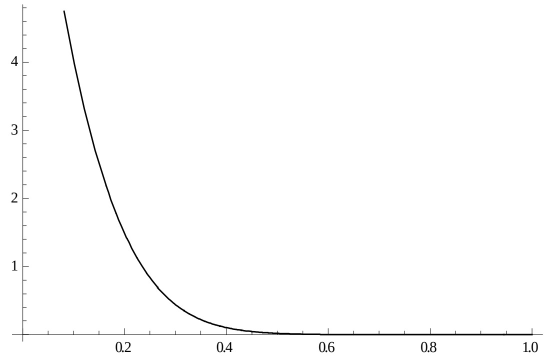

Fig. 2. The posterior distribution of p_1.

p_0	n	Model parameters $\lambda = 0.3$ $p_2 = 0.3$	 $\lambda = 0.3$ $p_2 = 0.8$	 $\lambda = 0.4$ $p_2 = 0.3$	 $\lambda = 0.4$ $p_2 = 0.8$	 $\lambda = 0.6$ $p_2 = 0.8$
0.2	100	0.1074	0.1028	0.0806	0.0833	0.0593
	200	0.0771	0.0806	0.0559	0.0616	0.0398
0.3	100	0.1270	0.1247	0.1103	0.1067	0.0756
	200	0.0963	0.0919	0.0569	0.0730	0.0534
0.4	100	0.1424	0.1309	0.0921	0.1073	0.0853
	200	0.0933	0.0953	0.0707	0.0718	0.0463

Table 3 The percentage of the acceptance of false H_0 for parameter p_1.

p_0	n	Model parameters $\lambda = 0.3$ $p_2 = 0.3$	 $\lambda = 0.3$ $p_2 = 0.8$	 $\lambda = 0.4$ $p_2 = 0.3$	 $\lambda = 0.4$ $p_2 = 0.8$	 $\lambda = 0.6$ $p_2 = 0.8$
0.2	100	0.1135	0.1268	0.0889	0.1081	0.0756
	200	0.0877	0.0960	0.0645	0.0756	0.0492
0.3	100	0.0948	0.1085	0.0711	0.0875	0.0588
	200	0.0728	0.0835	0.0698	0.0615	0.0435
0.4	100	0.0394	0.0404	0.0559	0.0482	0.0445
	200	0.0570	0.0549	0.0423	0.0522	0.0402

Table 4 The percentage of acceptance of false H_1 for parameter p_1.

As in the previously considered case, we assume that the uniform distribution on interval $(0, 0.5)$ is the prior distribution of p_1 and we made R=10000 repetitions to investigate the properties of the Bayesian test for hypotheses (12) and (13). The results of the simulation analysis for selected values of λ, p_2, p_0 and sample sizes are presented in the tables 3 and 4.

5. Conclusions

In this paper proposals of the Bayesian tests for a structural model are presented. We apply the Bayesian tests to verification hypothesis about model parameters.

The frequencies of false decision of the acceptance of H_0 or H_1 depend on the prior distribution of the considered parameter and the sample size. The application of the Bayesian tests requires large samples. In the cases analyzed, the frequencies of false decision are usually smaller than 10%. For smaller samples, the percentage of the decisions of the acceptance of false hypothesis is much bigger. In the case of the verification of hypothesis about structural parameter λ, and larger differences between parameters p_1 and p_2, the results were better. The frequency of the false decisions of the acceptance hypotheses about parameter p_1 also depends on the values of known parameters λ and p_1.

In a similar way, one can consider other types of the prior distribution of the model parameters and construct Bayesian tests.

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