The modeling of selected economic phenomena by means of positional game

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Abstract

Modeling economic phenomena, processes as well as a social interaction had been shaped over the years. The theory of mathematical models has own language, strictly specified form, depending on the nature of described issues. The game theory was firstly used for the description of economic phenomena in 1944's by John von Neuman and Oscar Morgenstern in: "Theory of Games and Economic Behavior". The game theory was turned out that to be an outstanding modeling device. But there are certain type of games, the so – called positional game with perfect information (Banach – Mazur games), which so far have not been applied in economy. The perfect information positional game is defined as the game during which at any time the choice is made by one of the players who is acquainted with the previous decision of his opponent. The game is run on the sequential basis. The aim of this paper is discuss selected Banach – Mazur games and to present some applications of positional game.

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1. Introduction

The most seriously played games of perfect information (which we call PI – games) are chess. Perfect information means that at each time only one of the players moves, that the game depends only their choices, they remember the past, and in principle they know all possible futures of the game. The first published paper devoted to general infinite PI – games is due to Gale and Stewart (1953), but the first interesting theoretical infinite PI – game was invented by S. Mazur about 1935 in the Scottish Book [7]. Positional games were created in 1940's by a prominent range of Polish mathematicians, belonging to the Lwow School of Mathematics. Owing to the authors' names they are otherwise known as Banach-Mazur games.

This paper aims to address the most common versions of Banach-Mazur games, their modifications and their possible applications.

2. The Banach - Mazur games and their applications

The relevant issue in the area of competitiveness is games displaying an infinite number of strategies. The overwhelming majority of dilemmas related to the above games were defined

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in the period from 1935-1941 and incorporated into the so-called Scottish Book. The Scottish Book referred to a notebook purchased by a wife of Stefan Banach and used by mathematicians of the Lwow School of Mathematics (such as Stanisław Mazur, Stanisław Ulam and Hugo Steinhaus) for jotting down mathematical problems meant to be solved. The Scottish Book used to be applied for almost six years. Many problems presented therein were created in previous years and not all of them were solved. After the World War II Łucja Banach brought the Book to Wrocław, where it was handwritten by Hugo Steinhaus and sent in 1956 to Los Alamos (USA, Mexico) to Stanisław Ulam. Ulam translated it into English, copied at his own expense and dispatched to a variety of universities. The book in question proved to enjoy such a great popularity that it was soon published and edited – mainly in English [1]. The Scottish Book presents the following game no. 43 elaborated by Stanisław Mazur [7].

Example 1. (Mazur)

Given is a set E of real numbers. A game between two players I and II is defined as follows: player I selects an arbitrary interval d_1, player II then selects an arbitrary segment (interval d_2 contained in d_1; then player I is turn selects an arbitrary segment d_3 contained in d_2, and so on. Player I wins if the intersection d_1, d_2, ...,d_n,...contains a point of set E; otherwise he loses. If E is complement of a set of first category, there exists a method through which player I can win; if E is a set of first category, there exists a method through which II will win.

Problem. It is true that there exists a method of winning for the player I only for those sets E whose complement is, in certain interval, of first category; similarly, does a method of win exist for II of E is set of first category (see [5])?

Addendum: Mazur’s conjecture is true.

Modifications of Mazur’s game are follows.

Example 2. (Ulam)

There is given a set of real numbers E. Players I and II give in turn the digits 0 or 1. I win if the number formed by these digits in given order (in the binary system) belongs to E. For which E does there exist a method of win for player I (player II)?

Example 3. (Banach)

There is given a set of real numbers *E*. The two players I and II in turn give real number which are positive and such that a player always gives a number smaller than the last one given. Player I win if the sum of the given series of numbers is an element of the set *E*. The same question as for example 2.

Example 4. (some popular modification Banach – Mazur game)

Two players choose alternatively one digit from the set 0, 1, …, 9. Through their choices they generate an infinite sequence of digits, e.g. 5791… Such a sequence may be denoted by a number $0.5791.. \in [0,1]$.. Before the game begins, a subset X of the section [0, 1] is to be defined. Player I win provided that the mutually generated number belongs to the set concerned. Player II wins if the number at issue does not fall within the set in question.
The conclusion seems inescapable that the above game has a winning strategy. One may assume that at the beginning the players should establish the set *X* taking the following form [0,1; 0,3]. Having arranged such a set, player I may initially select the digit 1 or 2, which strategy makes him win the game automatically. The selection of any other digit will result in the win of player II.

Formally we can write PI games as follows.

Let *A* is called the set of strategies of player I, *B* be the set of strategies of player II.

$\varphi: A \times B \to \overline{\Re}$, where $\overline{\Re} = \Re \cup \{-\infty, +\infty\}$ ($\Re$ is the set of real numbers).

This game is played as follows:

Player I chooses $a \in A$ and player II chooses $b \in B$. Both choose are made independently and without any knowledge about the choice of the other player. Then player II pays to I value $\varphi(a,b)$. If $\varphi(a,b) < 0$ means that II gets from I the value $|\varphi(a,b)|$.

Idea of an infinite game of perfect information is the following:

- let $\omega \in \{0,1,2....\}$,
- there is a set *P* called the set of choices,
- player I chooses $p_0 \in P$, next player II chooses $p_1 \in P$, than I chooses $p_2 \in P$, etc.

There is a function $f : P^{\omega} \to \overline{\Re}$, such that the end player II pays to I the value $f(p_0, p_1...)$.

Definition 1. The triple $\langle A, B, \varphi \rangle$ is said to be game of perfect information (PI – game) if there exists a set P such that A is set of all function:

$A = \left\{ a : \bigcup_{n<\omega} P^n \to P \right\}$, where $P^0 = \{\phi\}$,

$B = \left\{ b : \bigcup_{0<n<\omega} P^n \to P \right\}$,

and there exists a function $f : P^\omega \to \overline{\Re}$ such that $\varphi(a,b) = f(p_0, p_1 ...)$, where:

$$p_0 = a(\phi),\ p_1 = b(p_0),\ p_2 = a(p_1),\ p_3 = b(p_0, p_2),\ p_4 = a(p_1, p_3)\ .$$

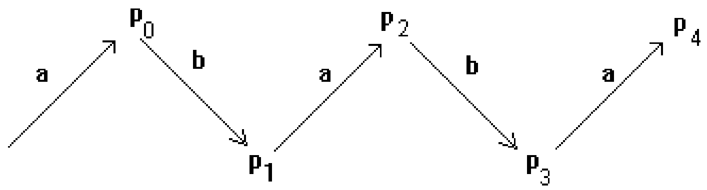

Fig. 1. PI – game.

A game $\langle A, B, \varphi \rangle$ defined in this way will be denoted $\langle P^\omega, f \rangle$ or $\langle P^\omega, X \rangle$.

The sequence $p = (p_0, p_1 ...)$ be called a game, any finite sequence $q = (p_0, ..., p_{n-1}) \in P^n$ is called position:

f is characteristic function of a set $X \subseteq P^\omega$,

$$\begin{cases} f(p) = 0 & gdy \quad p \notin X \\ f(p) = 1 & \text{gdy} \quad p \in X \end{cases} .$$

The player I wins the game if *f(p)=1* and II wins the game if *f(p)*=0.

Definition 2. [8]. A game $\langle A, B, \varphi \rangle$ is called *determined* if:

$$\inf_{b \in B} \sup_{a \in A} \varphi(a,b) = v = \sup_{a \in A} \inf_{b \in B} \varphi(a,b) \qquad (1)$$

where v is value of the game (common value *v* of both sides of this equation is called the value of the game $\langle A, B, \varphi \rangle$).

Remark. A game is determined if and only if the game has a value.

A game is not determined if :

$$\inf_{b\in B}\sup_{a\in A}\varphi(a,b) < v < \sup_{a\in A}\inf_{b\in B}\varphi(a,b)\,. \qquad (2)$$

Note. If the game is not determined, then the left – hand side of (1) is larger than the right-hand side of (1).

If the game has a value v and there exists an a_0 such that $\varphi(a_0,b)\geq v$ for all b, then a_0 is called an optimal strategy for player I. If $\varphi(a,b_0)\leq v$ for all a, the b_0 is called an optimal strategy for II.

We will say that $\left\langle P^{\omega}, f \right\rangle$ is a win for I or a win for the II if $\left\langle P^{\omega}, f \right\rangle$ has value 1 or 0, respectively. If $f : P^{\omega} \to \overline{\Re}$ has the property that there exists an n such that $f\,(p_0, p_1 ...)$ does not depend on the choice p_i with $i>n$, then $\left\langle P^{\omega}, f \right\rangle$ is called a finite game.

The following theorems are true.

Theorem 1. [8]. Every finite game has a value.

Proof ([8], proposition 2.1, pp. 45).

Theorem 2. [8]. There exist sets $X\subseteq\{0,1\}^{\omega}$ such that game $<\{0,1\}^{\omega},X>$ is not determined.

Proof ([8], proposition 3.1, pp. 46).

Theorem 3. [8]. If the set $X\subseteq P^{\omega}$ jest closed or open, then the game $\left\langle P^{\omega}, X \right\rangle$ is determined.

Proof ([8], proposition 3.2, pp. 46).

Theorem 4. [8]. If player II has a winning strategy in Banach - Mazur game, then X is not countable.

Another interpretations of Banach – Mazur games.

Example 5. (Mycielski)

A set S is given. Player I splits S on two parts. Player II chooses one of them. Again I splits the chosen part on two disjoint parts and II chooses one of them, etc. Player I wins if and only if intersection the chosen parts is not empty and player II wins if and only if it is empty.

Remark. Player I has a winning strategy if and only if $|S| \le 2^{\aleph_0}$, and the player II has a winning strategy if $|S| \le \aleph_0$, where $|S|$ means cardinality of set *S*, $\aleph_0$ is alef zero – cardinality of integer numbers.

Theorem 5. [8]. If player II has a winning strategy for Banach – Mazur game, then $|S| \le \aleph_0$.

The proofs of above theorems have used the Axiom of Choice [5]. Mycielski and Steinhaus conjecture that the Axiom of Choice is essential in any proof of the existence of sets $X \subseteq \{0,1\}^{\omega}$ such that the game $<\{0,1\}^{\omega}, X>$ is not determined. In the same order of ideas, theorem 5 shows that Continuum Hypothesis ($2^{\aleph_0} = c$ - continuum or we have not any cardinal number between $\aleph_0$ and $2^{\aleph_0}$) is equivalent to the determinacy of natural class of PI games.

3. Prime numbers and Banach-Mazur games

While creating the original variants of Banach - Mazur games, one may apply the properties of prime numbers, as they constitute a countable set. That ensures that the game in question may be deemed as determined.

Example 6. Game G_1. Player 1 chooses number $s_1=2n_1+k$, where $k<n$ and calculates an element of the sequence taking the form:

$$x_n = \begin{cases} 2^{-n} & \text{where} \quad \exists\ p,q \in \mathrm{P}_r \quad \text{such that} \quad 2n+k = p+q\ (*) \\ 2^{-k} & \text{where} \quad k \le n \quad \neg\exists\ p,q \in \mathrm{P}_r \text{ such that } 2n+k = p+q\ (*) \end{cases}, \tag{3}$$

P_r denotes the set of prime numbers whose divisor is 1.

Player II selects a subsequent number $s_2=2n_2+k$, where $s_2>s_1$ and finds an element of the sequence taking the form (3). The analogical action is taken by player 1, etc. If $\lim_{n\to\infty} x_n \to 0$, then player 1 wins; otherwise player 2 is a winner. Is there any winning strategy?

Note. According to [6] the properties of prime numbers may be summarized as follows:

Property 1. Each natural number bigger than 4 may be presented as the sum of two prime odd numbers.

Thus, player 2 can subsequently select odd numbers and in his k-step he may choose the number which fails to satisfy the condition (*). Then $x_k>x_k$-1, thus x_k does not converge to 0. Hence, player 2 has a winning strategy. It should be emphasized that the set of sequences taking the form (3) constitutes a set of first category and is countable. Therefore, referring back to the considerations as described in chapter 2, the game may be declared as determined (theorem 4 is satisfied).

Example 7. Game G_2. The game is analogical to game G_1 as defined in example 6. However, the numbers selected by the players in order to generate the elements of sequence x_n should satisfy the requirement (**) specified in the following formula:

$$x_n = \begin{cases} 2^{-n} & \text{where } \exists\, p,q,t \in \mathrm{P}_r \text{ such that } 2n+k = p+q+t \text{ (**)} \\ 2^{-k} & \text{where } k \le n \quad \neg\exists\, p,q,t \in \mathrm{P}_r \text{ such that } 2n+k = p+q+t \text{ (**)} \end{cases}. \tag{4}$$

Analogically to the previous case, P_r denotes the set of prime numbers. Is there a winning strategy?

In the event of G_2, the following property of prime numbers should be applied [6].

Property 2. Each odd number bigger than 7 may be presented as the sum of three prime numbers.

While applying property 2, one may assume that player II selects subsequent even numbers bigger than or equal to 8. Provided that, he may choose the number which fails to meet the condition (*). He has, thus, a winning strategy.

Conclusion . Games G_1 and G_2 are determined, since player II has a winning strategy.

4. Several remarks on positional games and its applications

Banach-Mazur games used to enjoy great popularity, mainly among mathematicians. When dealing with those games, the chief question was: is there a winning strategy guaranteed for any of the players? Taking into account the Axiom of Choice, already at the beginning of the 20th century it was proven that there were certain sets X for which neither player may adopt a winning strategy. The introduction of a new axiom to a set theory, known as the axiom of determination, significantly facilitated the search for a winning strategy. Different variants of Banach-Mazur games were analyzed in terms of the satisfaction of determination condition. It was proven that suppose one may find a set whose subsets are assigned a non-trivial measure, being a countable additive extension, vanishing on points and taking the value of 0 or 1, then all the analytic subsets defined on the set concerned are determined, or at least one of them has a winning strategy.

Banach-Mazur games can be classified as infinite multi-stage games with perfect information. In practice, they are illustrated by the situations where *the winner takes* *everything* (compare the Colonel Blotto Game [3]). Moreover, the games where the win is determined already at the initial stage rely on a *first come, first served* basis [3]. In terms of economy, such a game corresponds to the auction where a product (item) is offered up for bid. In such a case the buyer who wins the auction *takes everything* [2], [4]. Analogically to many positional games, the first participant submitting a bid determines the course of auction. Whenever the bid does not reach the sale price offered by the seller, other bidders may outbid the reserve price or withdraw from the auction. For instance, the digit selected by the participant initiating the game may not guarantee that the number generated in a following sequence will belong to a given interval (compare example 4). Notwithstanding the type of auction the optimal strategy adopted by a bidder resides in offering such a price which will warrant the win (i.e. the purchase of a product), however, which does not exceed his own valuations of an item in question. In the event of Dutch auction the price is gradually lowered until some auctioneer is willing to accept the announced price – such a participant wins the auction. It is a typical example of a game based on a *first come, first served* ground. The games introduced in examples 2 and 4 serve as an illustration for the Dutch auction [4].

The most common, “finite” positional game with perfect information is chess which laid foundations for artificial intelligence algorithms applied in various domains, including the construction of dynamic equilibrium models as well as the description of economic systems lacking the equilibrium.

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